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Mathematics Teachers' Geometric Thinking: A Case Study of In-service Teachers' Constructing, Conjecturing, and Exploring with Dynamic Geometry Software

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Abstract: Many research studies have been conducted on students' or pre-service teachers' geometric thinking, but there is a lack of studies investigating in-service teachers' geometric thinking. This paper presents a case study of two high school teachers who attended the dynamic geometry (DG) professional development project for three years. The project focused on the effective use of dynamic geometry software to improve students' geometry learning. The two teachers were interviewed using a task-based interview protocol about the relationship between two triangles. The interviews, including the teachers' work, were videotaped, transcribed, and analyzed based on the three levels of geometric thinking: recognition, analysis, and deduction. We found that the participating teachers manifested their geometric skills and thinking in constructing, exploring, and conjecturing in the DG environment. The study suggests that the DG environment provides an effective platform for examining teachers' geometric skills, and levels of geometric thinking and encourages inductive explorations and deductive skill development.

Keywords: *Dynamic geometry, geometric thinking, mathematics teachers.*

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Introduction

Making conjectures and writing proofs in geometry matches the practice standard of Common Core State Standards, which requires students to "construct viable arguments and critique the reasoning of others" (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2009, p. 6). However, writing proofs is a challenge to both students and teachers because of the difficulties involved in simultaneously reasoning about and depicting a geometric situation (Wares, 2004). Wares (2007, 2018) suggests that constructing geometric objects in a dynamic geometry environment is a meaningful and thought-provoking activity for many students. During this activity, students can make conjectures, talk about what they think, and work together to write a proof.

Other studies, such as Oxman et al. (2017), indicate that "working with the technological tool was successful for most of the students, and this activity raised their motivation to try and prove the hypothesis mathematically" (p. 615).

Dynamic software facilitates this process via the provision of visualization, manipulation, and justification (Goldenberg et al., 2008; Lopez-Real & Leung, 2006; Wares, 2004). Justification is a way to explain why construction works. Also, dynamic geometry software, such as GeoGebra allows students to create drawings, measure, and drag figures, and these activities may lead to conjectures and proofs (Mariotti, 2000). Additionally, such software's dynamic nature gives students the ability to have more options at hand to carry out their constructions with ease. Dynamic geometry software is widely recommended by teachers, teacher educators, and professional organizations for instructional contexts in which a dynamic geometry environment can assist with teaching geometry. "Using dynamic geometry software, students can quickly generate and explore a range of geometric examples" (Martin, 2000, p. 311).

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The challenge is to find a way to teach geometry to foster a more in-depth understanding. One suggestion is the use of dynamic technology that enables students to investigate geometric properties and, at the same time, examine many cases of geometric figures in the process of exploring, conjecturing, and possibly proving. Providing students with the opportunity to work with dynamic technology "extends their ability to formulate and explore hypotheses" (Martin, 2000, p. 310). Leikin and Grossman (2013) point out that "to perform a meaningful investigation in a geometry class, teachers should choose appropriate problems that facilitate experimentation, discovery, conjecturing, and the testing and proving of conjectures" (p. 517). The choice of such activities needs to be engaging, exciting, and open-ended to allow students to employ alternative problem-solving strategies. One way to facilitate rigorous, interactive learning is through the use of dynamic geometry software.

This paper is based on a larger study (Jiang et al., 2011), whose main goal is to investigate how the effective use of dynamic geometry software in secondary school mathematics teachers' classes may improve students' geometry learning. A great deal of research has been done on students' or pre-service teachers' geometric thinking (e.g., Armah et al., 2018; Hourigan & Leavy, 2017; Oxman et al., 2021; Wares, 2004, 2007, 2018), but there is a lack of studies investigating in-service teachers' geometric thinking.

In this study, the following research questions were investigated:

1. What strategies did the in-service teachers employ in their constructions using dynamic geometry software? What geometric skills and levels of thinking are related to these strategies?
2. How did dynamic geometry software help with the explorations and conjectures?
3. What geometric skills and levels of thinking are related to these explorations and conjectures?

Literature Review

Dynamic Geometry Software in Geometry Study

We believe that dynamic geometry software (or DGS) allows users to construct geometric objects and specify relationships. Previous studies have indicated the impact of using dynamic geometry on geometry teaching and learning (e.g., Christou et al., 2005; Oxman et al., 2021). Geometry has been traditionally taught with pencil and paper using lists of definitions, theorems, and proofs. The dynamic geometry approach fosters an environment that is more conducive to knowledge construction, exploration, conjecture, and proof (Furinghetti & Paola, 2003; Habre, 2009); promotes achievement in and the cultivation of a positive attitude toward mathematics (Phonguttha et al., 2009); and facilitates the development of reasoning and proof abilities (Jiang, 2002). More specifically, when using DGS in geometry learning and teaching, Abdelfatah (2011) indicates that using DGS was effective in enabling students to formulate and understand geometric statements, stimulating students to collaborate and share ideas, allowing the students to experiment and facilitate their grasp of the critical concept of proof, increasing students' interest in proof learning, facilitating the completion of proofs step by step, allowing the student to learn at his learning tempo, and increasing retention of proof learning and understanding.

Professional Development of Teachers in the Context of the DG Environment

Professional development offers a promising setting to support in-service teachers using integrated technology (Bennison & Goos, 2010). Technology, pedagogy, and content knowledge (TPACK) need to be emphasized in teachers' professional development (Healy, 2015; McBroom et al., 2016; Voogt et al., 2013, 2015). Dynamic geometry software allows teachers to create drawings quickly, accurately, and flexibly when teaching geometry (Nagy-Kondor, 2008; Oxman et al., 2017). Also, dynamic geometry software is a helpful tool to promote a teacher's instruction (Oxman et al., 2017).

Zengin (2017) discussed the effects of GeoGebra software on pre-service mathematics teachers' attitudes and views toward proof and proving and found that GeoGebra software was a useful tool to increase pre-service teachers' confidence about facilitating proofs and proving. Gueudet and Trouche (2011) conducted a study on training for secondary school teachers to foster an inquiry-based approach in mathematics teaching. This study focused on the investigative potential of dynamic geometry environments and found that dynamic geometry software contributed to the development of trainee practices.

Constructing, Exploring, and Conjecturing with DGS

The study by Crompton et al. (2018) investigated which technologies and technological affordances are specific to geometry learners. They found the following five main technical supports provided by DGS to geometry learners: visualization, manipulation, cognitive tools, discourse promoters, and ways of thinking. The promotion of TPACK was central to the study reported in this paper. Constructing a geometric figure permits many different strategies. Prior experience and knowledge about geometric figures' characteristics can serve as a segue to navigating iterative improvements to solution strategies. Jones (2000) states that construction "involves setting up construction and seeing if it is appropriate and quite probably having to adjust the construction to fit the specification of the problem" (p. 62).

"DGS (as a mediation tool) encouraged students to use (in problem solving and posing) the processes of modeling, conjecturing, experimenting, and generalizing" (Christou et al., 2005, p. 125).

In the context of dynamic geometry environments, to justify that a constructed figure exhibits desired invariant properties, the geometric figure must pass the *drag test* (Baccaglini-Frank & Mariotti, 2010). Drag-testing enables learners to explore geometric constructions in "visual-dynamic ways that could contribute to the formation of abstract knowledge and the bringing about of learning. Drag-testing empowers the users to objectify a personal continuum in which they might realize their geometric intuition in a dynamically real-time fashion" (Leung, 2008, p. 136). The drag-testing capability of dynamic software creates intellectually rich interactive scenarios and relationships instead of static paper and pencil environments (Goldenberg et al., 2008; Lopez-Real & Leung, 2006; Wares, 2004). Another essential tool of the dynamic software environment is its measuring capability. Users can measure "distances, lengths, perimeters, area, and angles of constructed figures […], and the measurement changes 'continuously'" (Olivero & Robutti, 2007, p. 137).

The Levels of Geometric Thinking with Geometric Skills

Hoffer developed a two-dimensional matrix to represent geometric thinking. The first dimension consists of five geometric skill areas (visual, verbal, drawing, logical, and applied). The second one deals with the levels of geometric thinking (recognition, observation, analysis, ordering, deduction, and abstraction) (Hoffer, 1981, p. 15). In the present paper, Hoffer's framework is simplified to include only the first four geometric skill areas (visual, verbal, drawing, and logical) and three levels of geometric thinking (recognition, analysis, and deduction). Since the two participants in the case had not completed the proof of the question, this study only focused on the processes of construction and conjecture. Two skill areas, drawing, and logic, and three levels of geometric thinking are included in the data analysis (see Table 1).

Table 1. The Levels of Geometric Thinking Distributed According to Geometric Skills

Skill area	Recognition	Analysis	Deduction
Drawing	Make sketches of figures accurately labeling given parts.	Transmit given verbal information to a picture. Use the given properties of figures to draw or construct the figures.	Recognize when and how to use auxiliary elements in a figure to support a conclusion. Deduce from given information on how to draw or construct a specific figure.
Logical	Realize there are differences and similarities among figures. Understand that shape is preserved when transformed by translation, rotation, and reflection, but not always by dragging.	Understand that figures can be classified according to different types. Realize that geometric properties can be used to distinguish figures.	Use logic to prove or disprove and be able to deduce new conclusions from given facts.

Methodology

Research Design

For this research, we opted for a qualitative approach. Both the goals of this study and the interpretive paradigm that underlies qualitative research are served by this approach. Participants for this qualitative study were chosen from teachers who were involved in a 3-year-long professional development project and engaged in activities that promote geometry teaching within a DG environment. The project participants were three university faculty members, three graduate students, and 40 high school geometry teachers randomly selected from three major school districts in one southern state in the USA. These school districts were involved in summer professional development to implement the DG approach in their classrooms. The teachers' teaching experience ranged from one year to 25 years of teaching high school geometry. Ten teachers had some experience using dynamic geometry software for teaching demonstrations, whereas most others did not.

Concerning the richness of interaction between the teachers, researchers, and technology, it should be borne in mind that out of the 40 geometry teachers who participated in the project, two of them (one male and one female) were selected for a case study (Creswell & Poth, 2016). The main reason for selecting a case study for this research is to deeply investigate the teachers' geometric thinking with dynamic geometry software. The teacher selection was based on the type of strategy used in constructing the required triangle in an arbitrary triangle ABC in each of the activities used in the summer professional development. These teachers were Lori and Ben†. Lori had been teaching Geometry for six years of her seven-year teaching career, whereas Ben had been teaching geometry for five years. The two participants had some experience using dynamic geometry software, but they only used it occasionally in their classrooms for demonstrations

†. The two-teacher names are pseudonyms.

in geometry construction. The construction strategies and their exploring and conjecturing are typical and representative, so Lori and Ben were selected for the task-based interview. We investigated how they explored and conjectured within the DG environment using task-based conversations.

The materials used in this study were:

(1) dynamic geometry software;

(2) A construction activity problem:

> *Construct an arbitrary triangle ABC, with points D, E, and F constructed on sides BC, CA, and AB, respectively, with segments BD = (1/3) BC, CE = (1/3)CA, and AF = (1/3)AB. Form triangle PQR using the intersections of line segments AD, BE, and CF. What is the relationship between triangles PQR and ABC?*
> and

(3) A task-based interview protocol about the relationship between triangles ABC and PQR.

The teachers were then presented with the construction activity problem. Participants were asked to construct triangles ABC and PQR (described) and to offer strategies to participants. Compared to other teachers' construction strategies, their construction strategies and responses to the questions asked by participants are typical and representative, so Lori and Ben were selected for case study research to participate in a task-based interview protocol. The interview protocol asked the two teachers to:

- Write down your initial conjecture about the relationship between these two triangles.
- Use the dynamic geometry software's measurement tools to take any measurements to check if the initial conjecture holds.
- Determine the relationship between the area of triangle PQR and the area of triangle ABC.

All of the sessions were videotaped and the teachers' handwritten work was collected.

Analyzing of Data

With the aid of dynamic geometry software, the two teachers presented their constructions of triangles ABC and PQR that were videotaped, transcribed, and analyzed based on three levels of geometric thinking: recognition, analysis, and deduction. Regarding their strategies of construction, the collected data included the following aspects:

- Teachers' first constructions: 1) Are the constructions correct? 2) Did they use a grid, calculation, or other techniques to construct the figures? 3) Did their first constructions pass the drag test?
- The improved constructions: 1) If their first constructions did not pass the drag test, how did they develop their constructions with DGS?

They were then invited to a task-based interview videotaped, transcribed, and analyzed based on the three levels of geometric thinking: recognition, analysis, and deduction. The interaction between the two teachers and the researchers proceeded with the task-based interview. The data were collected from the following aspects:

- The researchers asked the teachers to form an initial conjecture regarding the relationship between triangles ABC and PQR without using any measuring feature in DGS.
- The researchers asked the teachers to use measuring features in DGS to verify if their initial conjectures were correct or incorrect.
- They were asked how they formed reasonable conjectures with DGS if their initial conjectures were incorrect.
- The teachers were asked how they verified the conjectures with DGS.

Having thorough field notes, recording equipment, and a method to transcribe digital files helped ensure the consistency and accuracy of the data. Two researchers worked together to develop a coding method and reached a consensus on its use by reaching a high agreement rate. The researchers have extensive experience with qualitative research methodologies and teaching and studying geometry at the high school level.

Findings / Results

Teachers' Strategies for Constructing

One of the activities used in this summer's professional development was the construction problem used in this study:

> *Construct an arbitrary triangle ABC, with points D, E, and F constructed on sides BC, CA, and AB respectively, such that segments BD = (1/3) BC, CE = (1/3) CA, and AF = (1/3) AB. The triangle PQR is formed by the intersections of line segments AD, BE, and CF. What is the relationship between triangles PQR and ABC?*

The activity was performed after a week-long introduction to the DG approach that encompassed the use of DGS. The activity goal was to investigate various strategies teachers used to construct triangles ABC and PQR and establish a relationship between the two triangles and possible proofs of that relationship. Since this geometry problem is challenging, we did not expect that they would be able to prove it as part of the activity. Instead, this problem was used to solicit many conjectures that would eventually be proved false.

In this study, the researchers employed the ideas of construction operations defined by Mackrell (2011). Construction operations include creation, relative construction, transformation, measurement, tabulation, calculation, graphs, and algebra. Thus, the meaning of construction in this study does not necessarily mean "Euclidean construction" with straight-edge and compass only but instead refers to the use of electronic technology. The two teachers demonstrated their construction strategies and made conjectures using dynamic software.

Strategy 1 — Lori.

Lori employed the construction using features of dynamic geometry software as noted:

> Okay, I went ahead, and under the graph menu of DGS, I selected the *show grid* and then selected *snap points*. Then *make segments* that could easily divide by three, so I could get a third. So, I started with a right triangle (triangle ABC), then I just did a third of nine, and a third of six, then since this slope is up to six over nine, I went up two-over-three to get one-third. Once you have it up there, you can *hide the grid* and *snap points* and get rid of it [...], and it does pass the drag test.

Lori clicked and dragged the figure around to make sure that it passed the drag test: "If you measure this length (BD), and then this length (AF), it will always be one-third (see Figure 1). This is from a specific example; to get the generalized one is an excellent idea."

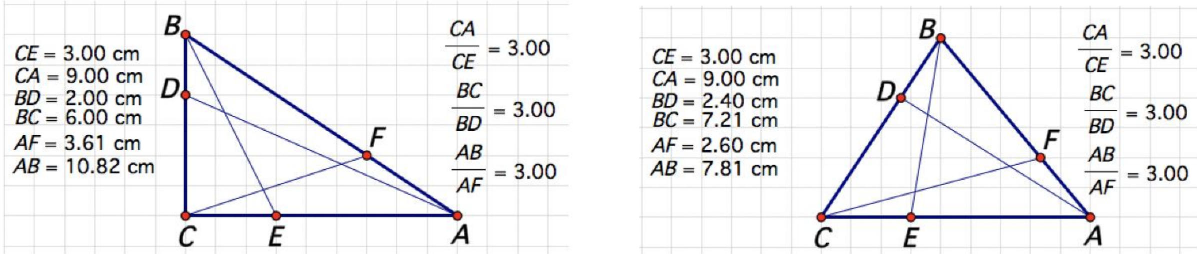

Figure 1. Two screenshots from Lori's dynamic geometry software construction, before and after the drag test

Lori's geometric skills and the levels of geometric thinking

Lori first sketched the figures accurately and labeled the given parts. That suggests her geometric thinking levels met the two levels: recognition and analysis. She also realized that the triangle is in the shape of a right triangle in the initial drawing, so she dragged it and transformed it into a triangle that looks more general. Based on her comments above, she concluded that the BD and AF ratio will always be one-third of the triangle proportionality theorem. This indicates the deduction level of geometric thinking.

Strategy 2 — Ben

In his description, Ben used the calculator tool to construct the 1/3 ratio for the three segments of triangle ABC, as described below.

> I started by getting my three points to make a random triangle; then, I created the segments. All right, this is when I started using the calculating tool. I wanted a segment that was a third of AB, so I measured the length of AB. Then, it gave me that length, but I want a third of that. So, I go over to the number and calculate it by clicking on AB divided by three. Now I got a segment length at least that is a third. What is cool about this program [...] click on a point, use a segment length to make a circle by a center and radius, and use any number. So, I constructed a circle with a center and radius; now, I have a point that is one-third of AB.

Ben did the same thing with (segment) BC, too. Then he went back to *calculate* and divided by 3 to create another circle. He did the same with AC, going on to *calculate* and then divide it by three, and hiding all of the circles to demonstrate that the construction would pass the drag test (see Figure 2).

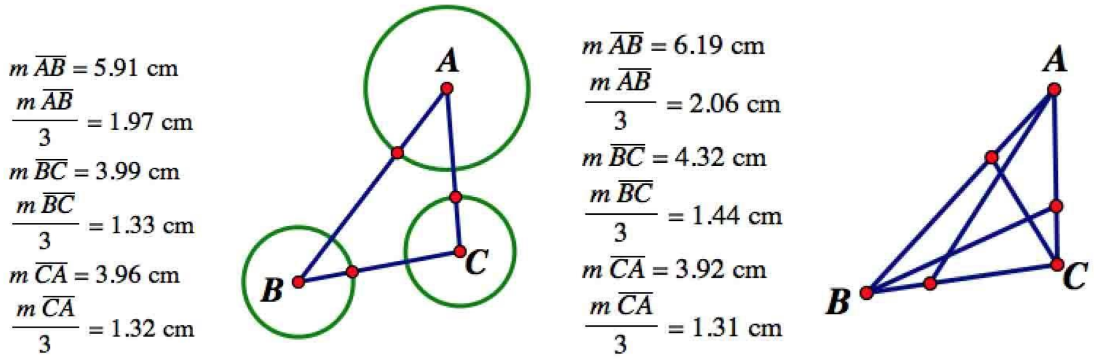

Figure 2. Ben's construction procedure

Ben's geometric skills and levels of thinking

Ben correctly sketched the figures and labeled the given parts completely with the measures of the relative segments. That suggests his geometric thinking levels met the two levels: recognition and analysis. No deducing was noted in the process of construction.

Teachers' Exploring and Conjecturing

To examine how teachers explored and conjectured, Lori and Ben were selected for the task-based interviews.

Case 1 — Lori

Lori was first presented with an interview protocol handout that had the pre-constructed figure (Figure 3), which was similar to what she constructed during her presentation. Before using the dynamic geometry software, Lori was asked to "look at the triangle (Figure 3) in the figure and write your initial conjecture about the relationship between triangle ABC and triangle PQR."

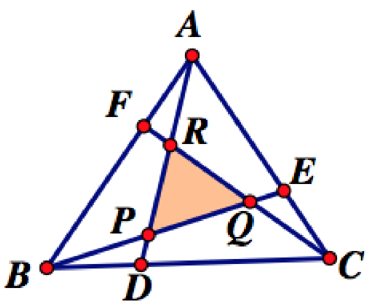

Figure 3. The Constructed Triangles ABC and RQP

Researcher:

We are trying to enter into conjecturing before we open the dynamic geometry software. Can you guess the initial conjecture regarding the relationship between triangles ABC and PQR?

Lori spent some time thinking about the relationship and concluded that the relationship triangle PQR is 1/3 the size of triangle ABC. Asked about why she thought that way, Lori noted that the segments had been divided into three equal parts used to construct the interior and would make the inner triangle 1/3 of the area of the original triangle. Lori was then asked to use the dynamic geometry software's measurement tool to take any measurements to see if her initial conjecture was correct and, if necessary, to revise the initial conjecture.

To investigate this, Lori opened the dynamic geometry software file that she had created and presented this to the group to investigate the conjecture. She then measured all the segments of triangle ABC and segments of triangle PQR (see Figure 4). She then constructed the triangle interiors of PQR and ABC to find their areas: "I conjectured that it would be approximately one-third." She then decided to investigate the area by finding the ratio of triangle PQR and ABC and noted that $\frac{Area\ of\ \Delta PQR}{Area\ of\ \Delta ABC}$ is $\frac{1}{7}$.

For further exploration, Lori measured segments BD, CE, and AF, but did not provide the researchers with any insights into the investigation. Lori assumed that the two triangles were similar and that the corresponding sides would be proportional. She tried the ratios of segments $\frac{RQ}{CA}$, $\frac{PR}{AB}$ and $\frac{QP}{BC}$ she did not get what she expected, which was that the two triangles would be similar and thus help prove the conjecture. She then measured the lengths of the big and small triangles to compare corresponding sides to tell if the two triangles were similar. In doing so, she took the ratio of the big triangle's longest side to the small triangle's longest side. She proceeded accordingly to the rest of the corresponding sides to test the Side-Side-Side similarity property:

> Okay, so I measured PQ, AB [...] from shortest to highest [...] from smallest to greatest. Nope, they are not similar. I thought they would be similar. [*Yelling at the screen.*] They are supposed to be similar! Aren't they? [*Laughs.*]

To verify the conjecture, Lori was focused on the triangle in the middle and the original triangle to figure out if they were similar.

$RQ = 2.18$ cm
$QP = 2.31$ cm
$PR = 2.08$ cm
Area $\triangle RPQ = 2.07$ cm^2
$m\,\overline{CA} = 5.56$ cm
$m\,\overline{AB} = 5.69$ cm
$m\,\overline{BC} = 6.14$ cm
Area $\triangle ACB = 14.46$ cm^2

$$\frac{m\,\overline{CA}}{RQ} = 2.54$$

$$\frac{RQ}{m\,\overline{CA}} = 0.39$$

$$\frac{PR}{m\,\overline{AB}} = 0.37$$

$$\frac{QP}{m\,\overline{BC}} = 0.38$$

$$\frac{\text{Area } \triangle RPQ}{\text{Area } \triangle ACB} = 0.14$$

$$\frac{\text{Area } \triangle ACB}{\text{Area } \triangle RPQ} = 7.00$$

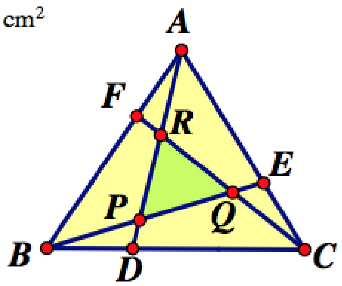

Figure 4. Lori's Measurements and Ratios Using the Dynamic Geometry Software to Try to Prove Similarity

Lori noted that triangles AFR, ECQ, and BPD have the same areas using the measurement tool. However, she still did not see anything that could help prove the relationship between triangles PQR and ABC by referring to this fact. Lori noted that the ratio of triangles RQP and ABC is 1:7, but that there was no similarity between the two triangles (see Figure 5). It was just the area that was one-seventh.

Lori was not sure how to start to prove the ratio of the areas as 1:7. She was aware that the triangle ABC segments were divided into the ratio one-third but was unsure how to use it. She tried using the measurement tool in the dynamic geometry software again but did not make much progress. To help Lori, the researcher gave hints to assist her.

Researcher: One thing that I found helpful is [...] dividing that quadrilateral [DPQC] into two triangles so you could use more triangle congruent/similarity, so you don't have to compare a triangle to a quadrilateral. But you can still do the quadrilateral way, but I thought it was a lot longer of proof, by constructing [a segment] from C to P.

Lori: Yeah.

Researcher: But it is your personal preference. I thought it was a bit shorter.

Lori: I don't want to do it. [*Laughs.*] I don't want to do a proof where the ratio is one-seventh.

To carry out the investigation, Lori constructed segments AQ, BR, and CP.

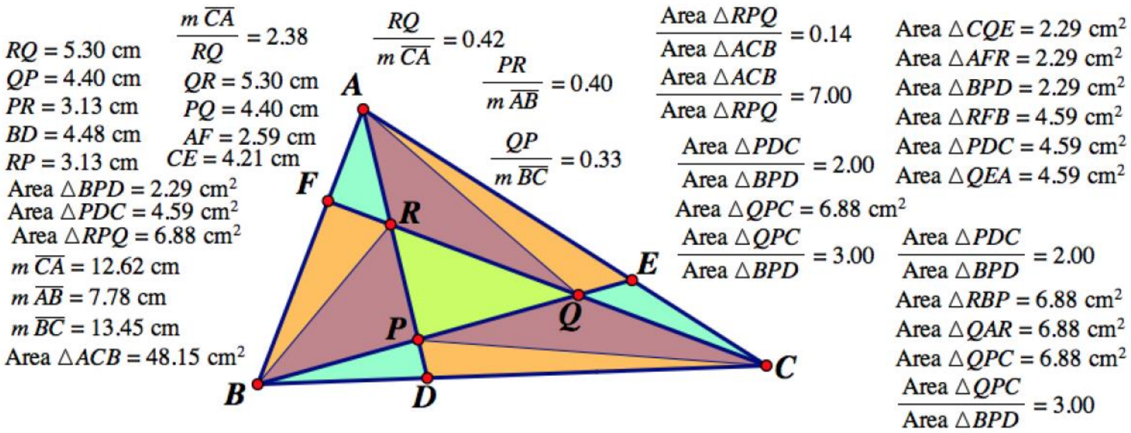

Figure 5. Lori's Measurements and Ratios Using the Dynamic Geometry Software

Lori looked at the figure to see if the triangles created were similar. To further challenge her thinking, the researcher engaged her in a discussion.

Researcher: What do you think would be the relationship between triangles BPD and CPD?

Lori: The relationship? All I know is that this [segment BD] is one-third of this [segment BC]. These are equal sides, so they are not similar.

Researcher: What about their areas?

Lori: proportional.

Based on the questions the researcher asked, Lori used the dynamic geometry software to measure the areas of triangles BPD and DPC and noted that the relationship was that triangle BPD was one-half of triangle DPC. Asked why that was the case, Lori could not see that both triangles had the same height but could note that segment DC was twice BD. Making a connection between the formula of finding the area of the triangle — (1/2)(base)(height) — and how that played into the area of the two triangles being twice the other was not evident in Lori's thinking. She was convinced from the dynamic geometry software investigation that the area of DPC was twice the area of BPD but could not explain why that was the case. Digging deeper into her thinking process, Lori had trouble proving why the two triangles (see Figure 6) had the area ratio 1:2. It was problematic for Lori to see that the two triangles had the same height, although she was aware that one had twice the base length.

Figure 6. Two Triangles Sharing the Same Height

Lori proceeded to use the dynamic geometry software and found that the triangle CPQ was three times the area of triangle BPD but could not prove why that was the case. At this point, Lori was unsure what to do, but the researcher asked her to investigate the relationship between triangle ABD and triangle ADC. She knew that triangle ABD had half the base of ADC but was not sure about the height since AD was not perpendicular to BC.

Lori: But those are not heights, though [...] so, I can't say that is the same altitude because it's not a 90-degree angle, so it's not even an altitude or height for the triangle.

Researcher: Which triangle are you looking at?

Lori: These two [referring to triangles ABD and ADC]. This one [ABD] and this one [ADC]. If I am going to use the fact that they have the same height, but the bases would be a 1:2 ratio, they are not accurate because they are not a 90-degree angle. They don't have the same height because that is not a 90-degree angle. This is not the common height of this triangle [ADC] and this triangle [ABD].

Researcher: Oh, I see what you are saying.

Lori: That means that the areas haven't been proven because that is not the same altitude, not the same height. It seems like that would need to be true [...] to use those ratios Area of ABD/Area of ADC = ½.

Researcher: Even if the altitude is exterior, it would still come to the base.

Lori: It's not exterior in this triangle [ADC].

Researcher: Are you referring to the green one [triangle PRQ]?

Lori: The [triangle] ADC.

Researcher: But it is still the same distance from [segment] BC to [vertex] A.

Lori: Okay. All right, so that's…

Lori was again having trouble understanding why triangle ABD and triangle ADC had the same height. She could not imagine the two triangles having the same height since the existing segment (AD) was not at a right angle to the base BC. The idea that triangle ADC had its height outside the figure was also very problematic to her. She imagined that the common height of the two triangles could be determined at the interior of the triangle, but in this case, such a scenario was not possible. Using the dynamic geometry software, the researcher demonstrated what the height might be in such a triangle. Based on that, Lori agreed about the ratio of the area of ABD/area of ADC = ½. Based on the discussion and the dynamic geometry software investigation, Lori came up with the following assumptions that let the area of BPD = x, so that the areas of the triangles then have the following relationship (see Figure 7):

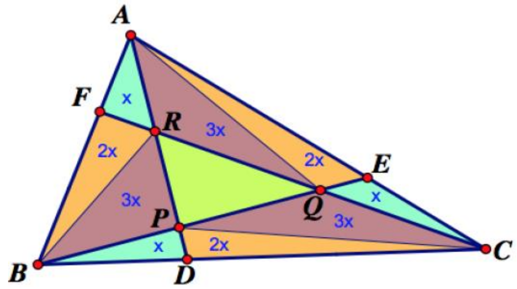

Figure 7. The Areas of the Triangles Relating to "x"

Lori was now faced with the challenge of showing that the area of ΔBPD = the area of ΔCEQ = the area of ΔAFR = x, meaning therefore that the area of ΔDPC = the area of ΔEQA = the area of ΔFBR = 2x which will lead to the area of ΔCPQ = the area of ΔARQ = the area of ΔBRP = 3x. Although some instances in the investigation were not proven, Lori noted that she could not have even attempted the investigation without the dynamic geometry software. The concept of the height of a triangle was an issue, especially when two triangles were side by side (ABD and ADC). Having height outside a triangle was confusing to Lori even though she knew it was a possible scenario. The Table below indicates Lori's logical skills and geometric thinking levels. We focused on logical skills (see Table 2).

Table 2. Lori's Logical Skill and Geometric Thinking Levels.

	Recognition	Analysis	Deduction
Logical	Had trouble determining the height of an obtuse triangle.	In analyzing the ratios of three sides between triangles ABC and PQR, Lori had the ratio of PQ and CA, which is not corresponding. PQ is the longest side of triangle PQR, but CA is not the longest one in Triangle ABC-this might indicate that Lori did not have clear ideas about the corresponding sides between two possible similar triangles.	Used logic to deduce new knowledge from given facts. For instance, the ratio of the area of ABD and area of ADC = ½.

Case 2 — Ben

Ben presented with an interview protocol that had figure 8, the same starting figure as Lori had. Ben asked to write down his conjecture about the relationship between triangle ABC and triangle PQR. Ben right away conjectured that the two triangles were similar. The ratio of their areas was 1:9 as he noted: "Since $\Delta ABC\ is\ an\ arbitrary\ triangle, and\ BF: BC = CD: CA = AE: AB = 1:3$, therefore the ratio of the two areas = 1:9."

He continued to state that if two shapes are similar with a scale factor of a/b, then the areas are in a ratio of $(a/b)^2$. Ben was very confident that triangle ABC and triangle PQR were similar. Still, he was not as sure about the ratio of their areas because he was unable to explain how the ratios BD = (1/3) BC, CE = (1/3) CA, and AF= (1/3)AB mentioned in the problem translated to the ratio of the sides or areas of triangle ABC and triangle PQR.

To investigate the two conjectures (the two triangles are similar and the ratio of their areas is 1:9), Ben opened the dynamic geometry software figure he had constructed (Figure 8). He then was asked to investigate his first conjecture that the two triangles are similar. He stated that "two triangles are similar if the corresponding interior angles are congruent and the lengths of the corresponding sides are proportional." Ben measured the triangle ABC and triangle PQR's interior angles and used the dynamic geometry software's click-and-drag test feature to investigate the corresponding angles. Ben realized that the two triangles were not similar and that therefore the initial conjecture was false.

Ben then went on to investigate his second conjecture that the ratio of the two areas was 1:9. Using the measuring feature in the dynamic geometry software, Ben measured the triangles ABC and PQR and found the ratio of the two areas. He then realized that the ratio of triangle ABC to PQR was 7:1. That was a bit of a surprise to Ben, and he noted, "This is why I like the dynamic geometry software; I could not have imagined that the ratio could be 1:7." Ben used the click-and-drag feature in the dynamic geometry software and noticed that the ratio remained constant (Figure 6 and Figure 9).

Asked why he was surprised about the 1:7 relationship, Ben said: "I know from my knowledge of geometry that if a length is increased by 3, the area increases by 9. Now that I divided the segment in the ratio 1:3, then it is natural that the area should be in the ratio 1:9."

Ben seemed to confuse this notion of the relationship between length and area about how the triangle was constructed. Ben made a mistaken connection between 1:3 and 1:9 because 1:3 is not the ratio of the corresponding sides between the triangles PQR and ABC, leading to a 1:9 ratio for the area of the two respective triangles.

Ben's connection contradicted his intuition since he noted earlier that triangle PQR and ABC are not similar. However, he noted the ratios of 1:3 and 1:9, contrary to the properties of similarity. The researchers did not find evidence that Ben used any deductive thinking to investigate 1:3, 1:9, and 1:7.

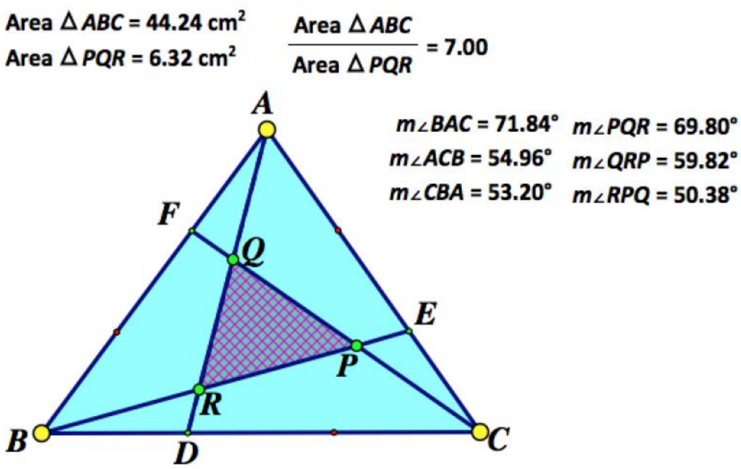

Figure 8. *Ben's Measurements and Ratios Using the Dynamic Geometry Software to Try to Prove Similarity*

The Table below indicates Ben's logical skills and geometric thinking levels. We focused on logical skills (see Table 3).

Table 3. Ben's Logical Skills and Geometric Thinking Levels

	Recognition	**Analysis**	**Deduction**
Logical	Understanding that shape preserves different properties.	From measures of interior angles of two angles, concluded the two triangles ABC and PQR are not similar.	None noted.

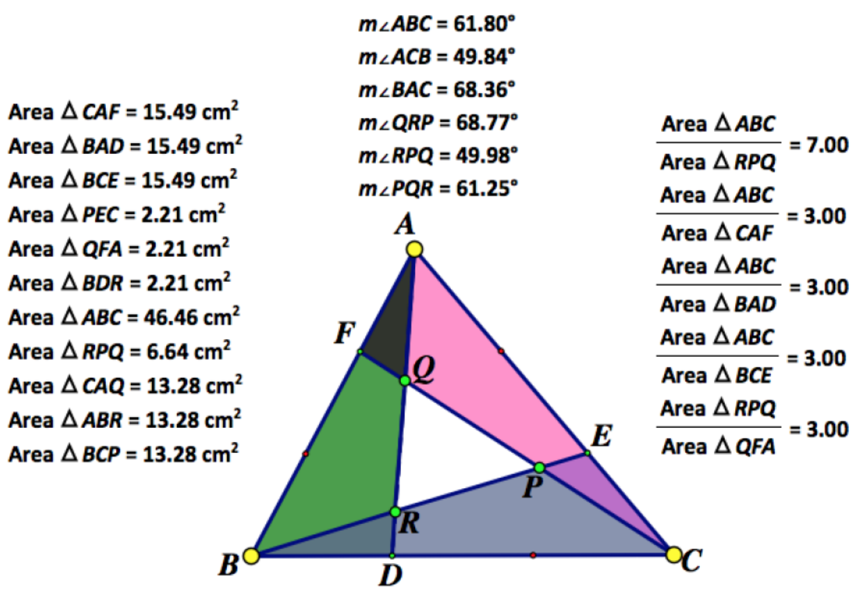

Figure 9. Ben's Additional Measurements and Ratios Using the Dynamic Geometry Software

Discussion

Knowing how to construct a figure is an integral part of the study of geometry. This is because one cannot construct a geometric figure unless one possesses an understanding of its features. Understanding the characteristics of geometric figures enables one to construct an accurate geometric figure and likewise determine if the construction is appropriate (Baccaglini-Frank & Mariotti, 2010; Wares, 2018). In short, to successfully construct a geometric figure, one needs to understand the problems presented. Additionally, as required by the problems, one must also understand the underlying geometric characteristics of what has been constructed.

Construction of the figure in the presented task was challenging because the participating teachers had to discover ways to create 1/3 of a side. The two participants used diverse strategies in their constructions and applied the drag feature in the software to verify the construction. We believe that the measurement tool's use was essential for checking to see if the constructions maintained the required characteristics, so the two teachers both finally correctly completed the drawing. The measurement tool in the dynamic geometry software plays a role in verifying a construction and passing the dragging test (Christou et al., 2004; Özen & Köse, 2013). The various strategies that the two teachers presented led to a discussion on the role of technology and on how teachers can use the feature to foster student discourse.

In this paper, the researchers provided evidence of how dynamic geometry software plays a role in helping teachers with the conjecturing process and in potentially guiding them to complete proof. In this case, the two teachers were first asked to write their initial conjectures about the relationship between those two triangles and test their conjectures using the dynamic geometry software. We regard this as the "phase preceding proof." As indicated in the literature, "The phase preceding proof helped students to build up empirical evidence for the plausibility of their conjectures" (Christou et al., 2004, p. 350).

As for the two teachers both received the reasonable conjecture, we think that the DG software provided not only an environment where the two teachers could investigate the constructed figure, but also a measuring and dynamic tool for them to formulate their conjectures, and find examples that disprove other conjectures. The measuring tool in DG software plays a significant role in facilitating the conjecturing process and making generalizations that cannot easily be observed without the dragging function (Özen & Köse, 2013). As seen in other studies, Ben and Lori easily conjectured and verified or disproved their conjectures using the measurement and dragging functions, which cannot be observed in the context of paper and pencil situations (Christou et al., 2004; Özen & Köse, 2013). As noted by Özen and Köse, dynamic geometry software "can be said to have a crucial role in problem solving, problem posing and using problem solving strategies" (p. 71).

This paper also provides evidence that dynamic geometry software, when used by motivated teachers and learners who are presented with a challenging problem, can help construct, explore, conjecture, and justify (whether true or false) conjectures. This study is consistent with teaching extreme triangle problems using dynamic investigation (Oxman et al., 2017). They found that the possibility of making frequent changes to the geometric objects and the ability to drag objects from the dynamic geometry software contributes to the process of deducing properties, checking hypotheses, and generalizing. Ben and Lori indicated that the DG environment facilitated their ability to construct, explore, and conjecture, all of which are processes that can lead to proof. As noted in this study, Lori initially found it challenging to understand how the areas of the two triangles were in the ratio of 1:2. Still, with the aid of the measurement tool, she was able to see the relationship.

Within the DG environment, the participating teachers manifested their geometric skills and thinking to construct, explore, and conjecture. Specifically, the two participating teachers employed different methods to construct the exact figure using dynamic geometry software. For the reasons why the two teachers used different methods of constructing the same figure, we argue that those methods are connected with the teachers' geometric skills and thinking and reflect their understanding of the dynamic geometry software's geometry and specific functions. Even though only two teachers were selected for the task-based interviews, it seemed that the teachers' exploring and conjecturing were related to their construction methods. For example, when the "drawing" skill of one of the interviewed teachers was coded with the geometric thinking level of "deduction," we found that this teacher did not solely rely on measurements to explore or conjecture. This teacher tried to construct a formal proof of the relationship between the two triangles' areas, which means that the geometric thinking level for this teacher's "logic" skill may reach the level of "deduction."

However, as for the other interviewed teacher, since the construction was made just by using the measurements, we concluded that the teacher's exploring or conjecturing relied entirely on measures and that the "logic" skill did not reach the "deduction" level of geometric thinking. In this regard, the DG environment may provide an effective platform for examining teachers' or students' geometric skills and geometric thinking levels.

This study suggests that working on challenging questions with dynamic geometry software and carrying out dynamic investigations can be a part of professional development for mathematics teachers.

Conclusion

Real-world problem solving, self-reflection on DGS-compatible metacognitive processes, and identifying potential implications are powerful learning opportunities. Since DGS's benefits won't happen spontaneously, teachers must carefully manage its effects. They must be developed through problem-solving contexts. There's little data on how problem-solving activities like DGS affect students' cognitive processing and problem-solving ability. How can instructors prevent DGS abuse? Investigating problem solvers who utilize DGS as a cognitive tool can give educators and researchers insights into what issue-solving requires, just as investigating teachers can give researchers and practitioners insights into how cognitive tools are successfully applied when solving problems. Math instructors can design effective solutions for a variety of school situations by focusing on this goal. This would allow the development of well-founded theories based on theoretical and empirical findings, thus adding to the mathematics education literature.

Recommendations

Although a DGS encouraged and helped some people find novel ways of thinking and applying them, for others, it stifled their natural ability to solve problems by encouraging them to become overly reliant on the DGS itself. Its applications vary depending on managerial judgment, resource management skills, and problem-solving experience. Using these results, I propose several possible approaches to teaching problem-solving skills that could benefit from further study and provide recommendations for future research in the field of technology-enhanced problem-solving.

Limitations

This study was conducted solely based on this case involving two high school mathematics teachers. More studies are needed to investigate DGS's effectiveness for other mathematics teachers with different backgrounds or beliefs about geometry instruction with DGS. Our next research study will examine how teachers' beliefs about geometry instruction with DGS impact their strategies for constructing, conjecturing, and exploring this software tool.

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