Mathematics Lecturers Professional Learning on the Topic of Even and Odd Functions through Lesson Study

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Abstract: A mathematics instructor with limited knowledge of content and pedagogy has little room for improvement or novelty in the classroom or the ability to arouse students' interest in learning mathematics. This case study was conducted in a foundation center of one of the public universities in Malaysia. The target of current research was to investigate the influence of lesson study (LS) on lecturers' pedagogical content and content knowledge. The LS group comprises of seven lecturers of the mathematics group and the researcher. The group collaboratively prepared a research lesson on the subject of even and odd functions. Data gathered through interviews and observations on the lecturers' activities in discussion meetings. Data from observations and interviews were analyzed descriptively and through thematic analysis method respectively. The results of this study show lecturers improved their knowledge in content and pedagogy considerably about even and odd functions. They enhanced their teaching knowledge through collaborative work and sharing of experiences. It seems the findings of this research not only help lecturers to have better performance in teaching the even and odd functions but also encourage them to experience the LS approach in teaching other mathematical concepts.

Keywords: *Content knowledge, even function, lesson study, odd function, pedagogical content knowledge.*

Introduction

Teachers' mathematical knowledge significantly influences on mathematics learning among students thus mathematics educators require to develop their teaching knowledge continually (Joanna O et al., 2018). Shulman (1987) divided the teacher knowledge into seven parts, namely "1) general pedagogical knowledge, 2) knowledge of learners and their characteristics, 3) knowledge of educational contexts, 4) knowledge of educational ends, purposes, and values, 5) content knowledge, 6) curriculum knowledge, and 7) pedagogical content knowledge" (p. 8). Teaching knowledge in mathematics explained as *"mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students"* (Ball et al., 2008, p.399). Although every skills and knowledge of mathematics educators is considered as a part of teaching knowledge, two components content knowledge and pedagogical content knowledge play an essential role to provide better performance among educators (Kleickmann et al., 2012). Based on Shulman (1986), mathematics content knowledge is the educator's knowledge of subjects matter taught. Ball et al. (2008) further extended the concept of content knowledge into two; common content knowledge (CCK) and specialized content knowledge (SCK). They defined CCK "as the mathematical knowledge and skill used in settings other than teaching" (p. 399). The CCK, the knowledge requires solving mathematics problems correctly through routine methods, and it is employed in various situations, not only in the process of teaching. Teachers apply their CCK using terms and notations in mathematics, and calculations. CCK assesses mathematical knowledge of educators that is not essentially related to teaching whereas, SCK is unique to teaching. Mathematics educators apply this category of knowledge to analyze students' errors in their solutions, their misunderstandings in mathematical contents, provide creative solutions for given problems, and mathematical procedures and algorithms. For instance, in the problem "solve the algebraic equation $|2x + 1| + |x - 3| = |3x - 2|$", teachers can discuss about the geometry method (drawing the graphs of functions $y = |2x + 1| + |x - 3|$ and $y = |3x - 2|$ to find the roots of this equation) or they can solve it using absolute value properties. These

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routine ways that need a lot of calculation are considered as CCK of mathematics teachers. Some teachers have better content knowledge to solve this problem by using SCK through the following creative method:

The roots of this equation will be the answers of inequality $(2x + 1)(x - 3) \geq 0$ so the answer is $(-\infty, -\frac{1}{2}] \cup [3, +\infty)$. Discussion about this creative method of solution in the class increases students' interest in learning mathematics conceptually.

Shulman (1987) defined pedagogical content knowledge as *"the blending of content and pedagogy into an understanding of how particular concepts, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8).* Ball et al. (2008) divided Shulman's PCK in to the knowledge of content and students (KCS) and the knowledge of content and teaching (KCT). KCS of educators is their knowledge of the interaction between learners and the mathematical materials. KCS domain helps educators to Know what mathematics topics or contents confuse, interest, or encourage students in learning mathematics (Ball et al., 2008). KCT of educators merges the knowledge of mathematical concepts and their teaching methods. KCT of educators is used when they decide what instructional method is suitable for a specific mathematics concept, and which examples should be used to engage the learners in mathematics learning. When educators decide to consider an exercise or a problem for students, they link their subject matter knowledge, pedagogy, and KCT (Ball et al., 2008). Bucat (2005) explained that *"There is a vast difference between knowing about a topic (content knowledge) and knowledge about the teaching and learning of that topic (pedagogical content knowledge)" (p. 2).* Therefore, pedagogical content knowledge points to knowing the content of mathematics and transfers it in the best way to students. Therefore, problem-solving is so important in mathematics classrooms in all educational levels.

A challenging mathematics task that the method of solution is not clear for students is called mathematics problem otherwise this task considered as a mathematics exercise (Schoenfeld, 1985). In fact, problem-solving refers to engaging in mathematics question that students have not before learnt how to find the answer. Importantly, the power of mathematics problem-solving among students enhance their skills in solving the variety of real world problems (Maasz & Schloeglmann, 2006; Tarmizi & Bayat, 2012). There is a strong positive relationship between mathematics achievement of students and problem-solving skills (Mon et al., 2016; Ramirez et al., 2016). In the traditional methods of teaching, usually learners memorize the mathematical methods and shortcuts, as a result they have serious problem in problem-solving because of superficial learning about the mathematics concepts (Khalid, 2017; Mon et al., 2016). Thus instructors require improving their teaching knowledge, especially about teaching methods, knowledge of content and pedagogy to encourage and motivate learners in learning mathematics. For example, routine methods of solution for the following problem require a lot of time and calculations but we can solve it easily by using the properties of even and odd functions.

Problem: Determine the value of $\int_{-2}^{2} x^3 \cos(x^2)\, dx$.

Routine solution: (Using integration by parts method)

$$\int x^3 \cos(x^2)\, dx = \int x^2 x \cos(x^2)\, dx$$

$$u = x^2 \text{ and } dv = x \cos(x^2)\, dx$$

$$du = 2x dx \text{ and } v = \left(\frac{1}{2}\right) \sin(x^2)$$

$$\int x^3 \cos(x^2)\, dx = x^2 \left(\frac{1}{2}\right) \sin(x^2) - \int \left(\frac{1}{2}\right) \sin(x^2)(2x) dx$$

$$= \left(\frac{1}{2}\right) x^2 \sin(x^2) - \int x \sin(x^2)\, dx$$

$$= \left(\frac{1}{2}\right) x^2 \sin(x^2) - \left(\frac{1}{2}\right)\left(-\cos(x^2)\right) + c$$

$$= \left(\frac{1}{2}\right) x^2 \sin(x^2) + \left(\frac{1}{2}\right) \cos(x^2) + c.$$

Therefore,

$$\int_{-2}^{2} x^3 \cos(x^2) dx = \left(\frac{1}{2} 2^2 \sin(2^2) + \frac{1}{2} \cos(2^2)\right) - \left(\frac{1}{2}(-2)^2 \sin(-2)^2 + \frac{1}{2}\cos(-2)^2\right)$$

$$= \left(2\sin 4 + \frac{1}{2}\cos 4\right) - \left(2\sin 4 + \frac{1}{2}\cos 4\right) = 0.$$

Mathematics educators with appropriate KCT can help students to solve this problem easily based on the following theorem.

Theorem: Assume that the function f is continuous on $[-a, a]$ then:

    i.   For the even function f we have $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

    ii.   For the odd function f we have $\int_{-a}^{a} f(x)dx = 0$

Creative solution:

Based on the above theorem, the value of $\int_{-2}^{2} x^3 \cos(x^2)dx$ is zero because $y = x^3 \cos(x^2)$ is an odd function.

This innovative solution shows an application of even and odd functions in integration concept. Suitable mathematical materials and pedagogy on the topic of even and odd functions provide better performance among mathematics educators. To achieve this goal, educators in the context based professional learning programs can work collaboratively to prepare mathematics lessons and materials based on students' abilities in problem-solving in order to emphasize on mathematical problem-solving in their teaching (Baker, 2021). Professional development programs are so essential for mathematics educators to improve their teaching quality and practice. Based on Avalos (2011, p.10) *"professional development is about teachers' learning, learning how to learn, and transforming their knowledge into practice for the benefit of their students' growth. Teacher professional learning is a complex process, which requires cognitive and emotional involvement of teachers individually, and collectively the capacity and willingness to examine where each one stands in terms of convections and beliefs and the personal and enactment of appropriate alternatives for improvement or change".* All mathematics educators' behaviors in managing the mathematical activities in classrooms are considered as part of professional development programs (Rose & Reynolds, 2014). They further added that mathematics educators can improve their teaching knowledge through professional development by the following methods:

    i)   Direct teaching such as courses, workshops and seminars

    ii)   Teaching in school such as action research and collaborative work with other educators

    iii)   Out of school learning such as e-books and visit to other educational centers

The mathematical function is a key content in curricula of mathematics (Akkus et al., 2008; Ponce, 2007). For example, two concepts composite function and inverse function are considered as challenging and conceptual topics among educators to transfer to learners (Oehrtman & Carlson, 2008). Modeling the real-world problems in different areas of human life is the most common application of the mathematics functions (Michelsen, 2006). In some mathematics classrooms, teachers teach the function concepts to students through superficial teaching methods because of their weaknesses in subject matter knowledge and pedagogy (Gholami, 2022). Most of educators are not familiar with problem-solving approach, thus they need to improve their content knowledge and pedagogical knowledge to have suitable performance (Gholami, 2021). It seems, this research helps educators to enhance their teaching knowledge on the topic of even and odd functions through collaborative work.

Lesson Study

LS approach in teaching mathematics was used in Japan from long time ago (since the middle of 20th century) as an educators' professional development program (Abiko, 2011). This effective professional development focuses on two important characteristics in education namely mathematics content and students' learning (Coenders & Verhoef, 2019; Penuel et al., 2007). LS is a new method of teaching that a group of educators through collaborative work, prepare an effective lesson and discuss on the taught lesson in order to enhance the ability of students in learning mathematics based on problem-solving approach (Matanluk et al., 2013). These lessons are named study lessons (Yoshida, 1999) or research lessons (Fujii, 2016). In the cycle of LS, research lesson is the central factor (McDonald, 2009). The main target of LS in teaching mathematics is improvement in teaching quality through a collaborative process among educators (Dudley, 2011). Fujii (2014) introduced a model for LS with five steps as below:

    a. *Goal setting*: In the first step, educators consider long-term targets to enhance the situation of mathematics learning among learners.

    b. *Lesson planning*: In this phase, educators through collaborative work design a research lesson with appropriate mathematical materials to improve the skills of learners in problem-solving.

    c. *Research lesson*: A member of the LS group teaches the prepared research lesson and other members of the group observe the taught research lesson and gather data to enhance the quality of it.

    d. *Post-lesson discussion*: In this part, educators try to enrich the research lesson according to the factors, skills of students, learners' misunderstandings, and variety of solutions for a given problem.

    e. *Reflection*: In the last phase of this process, educators discuss to prepare some new mathematical problems for the research lesson and solve them in the next LS cycle. Finally, they provide a report on the prepared research lesson.

Even and Odd Functions

The function f with the property $f(-x) = f(x)$ for all values of x in its domain, is called an even function. Based on this definition, even functions are symmetric across the y-axis. In other words, both points (x, y) and $(-x, y)$ are on the graph of function. For instance, $f(x) = x^2 \cos x$ is an even function. Similarly, the function f is considered as an odd function if $f(-x) = -f(x)$ for all values of its domain. It means the graph of odd function is rotationally symmetric around $(0, 0)$ and both points (x, y) and $(-x, -y)$ are on the function graph. For example, the Sign Function is an odd function. The rule of Sign Function is as follows.

$$f(x) = sgn\,(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

This function can also be written using the floor and the absolute value functions as

$$sgn\,(x) = \left\lfloor \frac{x}{|x|+1} \right\rfloor - \left\lfloor \frac{-x}{|-x|+1} \right\rfloor.$$

Thus based on the definition of odd function,

$$\forall x \in R, sgn\,(-x) = \left\lfloor \frac{-x}{|-x|+1} \right\rfloor - \left\lfloor \frac{x}{|x|+1} \right\rfloor = -\left(\left\lfloor \frac{x}{|x|+1} \right\rfloor - \left\lfloor \frac{-x}{|-x|+1} \right\rfloor \right) = -sgn\,(x).$$

Each function is even, odd, both, or neither. For instance, the function $f(x) = x^3 + 5$ is neither even nor odd.

Theoretical Framework

The LS model in mathematics education needs to be supported with a strong theoretical foundation. Constructivism from a social constructivist perspective helps create a platform that assists the use of LS as a potential approach for improving the skills of lecturers in mathematics teaching. The application of constructivist principles prepares an opportunity for lecturers to have better perspectives about mathematics materials in research lessons and helps improve their proficiency in transferring the materials to students through problem-solving approach. Hoover (1996) believed that constructivism provides a guide to the educators on how to support students construct knowledge in mathematics instead of merely transferring knowledge from the lecturer to the learners. Also, Hoover explained that lecturers who prepare their lessons based on constructivist principles make use of the learners' prior mathematics knowledge in providing learning situations that minimize inconsistencies between the learners' existing and new knowledge to be constructed in order to manage the time of teaching effectively. Furthermore, Hoover had emphasized the need for lecturers to engage learners in learning and to get them involved actively in the process of learning by using suitable materials and methods of transferring through problem-solving activities. By focusing on LS in teaching process, not only mathematics educators improve their performance through collaborative work and sharing their experiences but also students learn mathematics deeply by improving their problem-solving power (Huang & Shimizu, 2016).

Methodology

Research Design

In Malaysian education system, students get the Malaysian Certificate of Education examination at the last year of high school (Form 5). After that they have the variety of choices to continue their studies in a pre-university program. They can enroll in Form 6 or Matriculation that run by the Ministry of Education, A-level or Foundation program which conducted by selected universities. Their selection in each pre-university program is according to the performance of them in the selected subjects such as Additional Mathematics, Mathematics, Physics, and Chemistry. The researcher conducted this qualitative case study at the foundation center of a public university in Malaysia. The target of current study was to enhance the knowledge of content and pedagogy of lecturers about the topic of even and odd functions in pre-university level in order to enhance the quality of teaching among lecturers. This foundation center was selected for this study, because it provides a conducive and supportive place for the LS to be implemented.

The foundation program lasts one year (two semesters) and lecturers teach two textbooks Mathematics 1 and Mathematics 2 during this educational program. The topic on even and odd functions of Mathematics 1 textbook was

chosen because the lecturers identified it as a problematic concept for the students. The LS group then through collaborative work, planned, discussed, and designed a research lesson on the topic of even and odd functions. Two weeks before starting this research, the researcher introduced the LS to lecturers and asked them to prepare their materials about even and odd functions using Mathematics 1 textbook, online webs and other existed sources. In two discussion sessions, lecturers collaboratively planned, discussed and designed a research lesson on the topic of even and odd functions. During this period of time, they shared their content knowledge and experiences on the teaching of this topic. The researcher not only managed the discussion meetings but also participated in discussion meetings as a member of LS group.

The production details of this research lesson were according to the LS model of Fujii (2014), in fact, the LS group members prepared a research lesson on the topic of even and odd functions through the following phases:

Goal setting: In this phase, the researcher explained to the lecturers to prepare a research lesson that help lecturers to transfer this concept better to students. Therefore, they focused on mathematical content knowledge and pedagogy about even and odd functions according their experiences in teaching.

Lesson planning: In a discussion meeting, lecturers through collaborative work designed a research lesson about the even and odd functions with suitable mathematical materials to enhance the skills of learners in problem-solving.

Research lesson: A member of LS group that was chosen randomly taught the prepared research lesson in a real existing class contains 42 students and other members of group observed his teaching and collected data in order to discuss regarding the teaching of different parts of this research lesson.

Post-lesson discussion: In a meeting, lecturers tried to enrich the research lesson based on data they collected in the teaching session. For example, some lecturers believed that after discussing regarding the definitions of even and odd functions they should engage students with the task "give an example for even, odd, both even and odd, and neither even nor odd function" because discussing about this example provide students to learn conceptually the concept of even and odd functions. As another example in this phase, the LS group members removed the following task since they argued that this problem is too hard for students.

"A linear function intersects the x-axis and y-axis in h and k respectively.

a) State the value of k for which $y = f(x) + k$ is an odd function.

b) Find the value of h for which $y = |f(x + h)|$ is an even function."

Reflection: Finally, lecturers prepared the final report for this research lesson. Furthermore, the researcher explained to lecturers that they can add, remove or enhance the mathematical materials of this research lesson to improve the quality of it in the future if they plan to use or share this output.

In this research, data gathered through interviews and observations the lecturers' discussions. The researcher and two mathematics education experts was prepared interview protocol about pedagogical content knowledge and content knowledge of lecturers and sent these questions to three mathematics education experts in Malaysia, Turkey, and North Cyprus to ascertain the validity and suitability of them. After the experts' comments on the interview questions were applied, some experts in the Ethical Committee of the Research Management Center (RMC) of the same university were confirmed the final version of this interview protocols. Two questions of interviews are "What is your experience about collaborative learning environments regarding the even and odd functions?" and "What is the best advantage of LS in mathematics teaching based on your experience in this research?". In this study, the researcher recorded the interviews with all lecturers using an audio recorder.

Participants

In this foundation center, there were nine mathematics lecturers (four males and five females). The purpose and process of this study was explained for lecturers face to face. Firstly, eight mathematics lecturers have agreed to participate in this research, but later, one of them has withdrawn from participating in because of his time constraint. Therefore, the LS group of this research involved the researcher and seven lecturers of mathematics unit. The researcher had several roles in this study such as the coordinator, discussion leader, and a member of LS group. After permission of this study was obtained from the Principal of the Foundation Center, all lecturers and students who were a part of this research were asked to sign the letter of consent. The biodata of participants in this study was shown by Table 1.

Table 1. Biodata of the Study Participants

Lecturer	Age (Year)	Sex	Degree	Teaching Experience (Year)
A	28	Female	Master	2
B	28	Female	Master	3
C	33	Male	PhD	4
D	54	Female	Master	20
E	32	Male	PhD	1
F	28	Female	Master	3
G	47	Male	Master	10

Data Analysis

In this study, the researcher tried to collect data as truthful and accurate as possible to obtain better results. These methods not only reduced errors of this qualitative research but also prepared a vast opportunity for the researcher to be familiar with the individual ideas of lecturers and their performance in the group discussions. The researcher is a mathematics educator outside of this foundation center, so the participants in this study answered the interview questions without concern because there were no sources of bias that would negatively affect the quality of the data. The gathered data from observation of discussions among lecturers in the meetings were analyzed descriptively. The analyzed materials that provided by lecturers in discussion meetings sent to three mathematics experts in a Malaysian public university for content review and compare with materials in the textbook. They commented that the content of this research lesson is not only much richer than the content of the textbook, but also suitable for foundation level students and has no scientific problems. The collected qualitative data through interviews with lecturers was analyzed using the thematic analysis method. The researcher examined the qualitative data carefully to clarify ideas and patterns of meaning that appear repeatedly in the interviews. Therefore, six phases namely, cognition, coding, producing themes, checking themes, defining and naming themes, and writing up report were considered in the analyzing process of this study to find the general themes (Kiger & Varpio, 2020).

Findings

The findings of lecturers' content and pedagogical content knowledge on the topic of even and odd functions are studied through interviews and observations. Two methods were applied to triangulate the findings on the lecturers' knowledge in content and pedagogy.

Lecturers' Content and Pedagogical Content Knowledge as Identified Through Interviews

The qualitative data of interview transcripts was analyzed through the thematic analysis method (Braun & Clarke, 2006; Kiger & Varpio, 2020), which contained an repetitive cycle that moved among data, theory, and interpretation. This part of findings, discussed based on six main themes namely, subject matter knowledge, teaching time, assessment, teaching materials, teaching method and student activity. The summary of lecturers' ideas on the teaching of even and odd functions, after being involved in the LS program discussed as follows.

Subject Matter Knowledge

All lecturers said that LS program was enjoyable for them because during this study they learned a lot of new things in teaching the mathematical concepts of even and odd functions, thus their subject matter knowledge improved considerably. Four lecturers C, D, E and F described, discussion about different mathematics problems with a lot of solutions were so useful to improve their content knowledge. In this research, the process of studying even and odd functions discussed in a wider mathematical domain to show that what students learn while studying these types of functions could be applied to other mathematical areas such as integration concept and domain of functions. Therefore, this view of mathematics teaching improved the content knowledge of lecturers in the areas of integration, domain of functions, and even and odd functions. For example, lecturers suggested solving the problem "find the range of function $y = ||x| - 2|$" by drawing the graph of this function. In this study, they found that how to apply the properties of even function in solving this problem. The function $y = ||x| - 2|$ is an even function and the y-axis is the symmetrical axis of this function. Therefore, it is enough to find the range of this function only for non-negative real numbers. If $x \geq 0$ then $|x| = x$ and $y = |x - 2|$. It is clear that the range of this function is $[0, +\infty[$. This method of solution was new for the majority of lecturers. Lecturer B explained that:

> In this program, I learned many new beneficial things about the algebraic combinations of two or more even and odd functions such as $f \pm g$, fg and $fg + g$. Logically proofs for problems on even and odd functions help me to have better performance in my teaching. In fact, these mathematical arguments emphasize on the theoretical knowledge behind the concept of even and odd functions thus my content knowledge improved. She further added I have already confirmed the problems related to the combinations of even and odd functions

with a few examples, but this study helped me to know the logical argument for these problems should be discussed in the teaching in order to enhance the students' critical thinking.

LS is a strong program of professional development that prepares vast opportunity for mathematics educators to find their weaknesses and misunderstandings about mathematical concepts (Mhakure, 2019). In other words, the SCK of lecturers enhanced during this research. Lecturer A said that:

In LS program, my content knowledge enhanced because teaching method focuses on deeply understanding the mathematics materials through problem-solving approach. I have a few year experiences in teaching, this study was useful for me to learn some concepts on the even and odd functions conceptually. In fact, LS through discussion and studying improved my content knowledge. For example, before of this program, I taught only the conditions $f(-x) = f(x)$ and $f(-x) = -f(x)$ to determine even and odd functions respectively. But later in a discussion meeting I found that the domain of function is so important to be considered in determining the even and odd functions. For example, based on my previous understanding, the function $f(x) = \sqrt{x} + \sqrt{-x}$ satisfies in the condition $f(-x) = \sqrt{-x} + \sqrt{-(-x)} = \sqrt{-x} + \sqrt{x} = \sqrt{x} + \sqrt{-x} = f(x)$ thus this function is an even function. Now, I understand this function is both even and odd, because the domain of this function is $D_f = \{0\}$ and $f = \{(0,0)\}$. Therefore, to clarify the even and odd functions, firstly the domain of function should be symmetric with respect to the origin and secondly we should test the conditions of even and odd functions.

Also, Lecturer D clarified that:

When I was a student in high school and college, I just memorized a lot of difficult mathematical concepts without understanding them. Teaching helps me to have a very deep understanding of the textbook materials, although I still have difficulty in understanding some of the mathematical concepts. Therefore, I try to improve my mathematical knowledge to provide mathematics topics to students in attractive and understandable ways.

Teaching time

Lecturers A and B described although through LS method, lecturers can manage the time, it is a big challenge for problem-solving approach because a lot of sub topics should be covered. Other five lecturers explained that they can design appropriate research lesson for students to learn faster and easier the mathematics materials through better pedagogy. Despite, time is a challenge for mathematics educators to prepare a research lesson outside the class (Kanellopoulou & Darra, 2019), suitable research lesson helps lecturers to manage the time without difficulty inside the class. They clarified an important note that in LS, they can send the research lesson to students through Putrablast so technology reduces the time of students copying down notes. For instance, although lecturers spend two sessions to prepare a research lesson on the topic of even and odd functions and they participated one session in a real class they can use this prepared research lesson in the future if they want to use LS. In fact, they can improve this research lesson through informal discussions and no need to spend a lot of time for this issue. Lecturer E explained that:

In traditional teaching method, approximately 50% of teaching time allocates to students copying down notes in the classroom and I consider only a few mathematics exercises for students to solve in each session. In LS approach, the time for copying down notes reduces to about 15% because in this approach lecturers send the lessons to students before starting the class. It means LS not only provides a considerable time for problem-solving activities in each session but also improve the educators' knowledge especially in content and pedagogy. In fact, in this research lesson on the topic of even and odd functions, students engage with some activities that reduce the time of copying down notes among students.

Assessment

Lecturers before participating in this program explained that they are assessing students through two tests, two quizzes and a final exam based on the schedule of foundation center. At the end of this research, all lecturers were agreeing that they should give students some activities and problems in each session to check their abilities and skills. Lecturers clarified that based on LS program they will assess students using some given problems in each session to be solved individually and in team to enhance their abilities in mathematics learning. Assessing students' ability is an important goals of the LS, so in the post-lesson discussion stage, educators discuss variety of solutions, students' weaknesses, and possible misunderstandings in solving the given problems, in order to have better performance in students' assessment (Fujii, 2016). All lecturers except one of them believed that they prefer to design the exam questions teamwork because through collaborative work and discussion with their colleagues, they have better output in formulating questions that are suitable to assess students. Meanwhile, in LS, mathematics educators provide some mathematical activities to measure the ability of students in problem-solving. Lecturer G clarified that:

In this program, I found that students' assessment is an important part of teaching. In order to have an active teaching, I will choose randomly some students and ask them some questions. I should measure students understanding through presentation, questioning and communication because some students memorize mathematical materials and apply them in exams. Also, the item difficulty of exam questions must be in such

level that measures the ability of students in problem-solving and critical thinking. Therefore, preparing exam questions by a group of lecturers is better than individually work. For example, based on my experience, usually lecturers consider some questions such as "show that the function $f(x) = x\sqrt{9 - x^2}$ is an odd function" about the definition of even and odd functions in their exams. Some lecturers in this research believed that instead of these routine questions we should use some conceptual questions such as "prove that product of two odd functions is an even function" or "how many both even and odd functions can we find?" in the exams in order to improve the ability of students. Because mostly the exam questions are routine and students only focus on closed questions to previous exam questions to obtain a good result. If students know some of the exam questions need to critical thinking, definitely they learn the mathematical concepts deeply and improve their ability in problem-solving.

Teaching materials

The lecturers said in LS they have daily lesson plan according to the research lesson. They share their knowledge and ideas with other lecturers in order to improving the research lesson by adding some new methods and materials and correct some likely mistakes (Takahashi & McDougal, 2016). Lecturer C said:

The members of LS group emphasis on the theory behind the mathematical concepts and formulas instead just simply introduce the main concepts and formulas to students through memorization method. Also, discussion on the relation between different topics is so useful for lecturers to prepare suitable materials in the research lessons. For example, in this study, I saw the application of even and odd functions in the topic of range of function which was new for me.

All lecturers explained that by using LS approach, they can find their weaknesses and strengths about even and odd functions and by collaborative work with their colleagues; they can share their experiences and knowledge. Thus this collaborative work improved their individual daily lesson plan. Lecturer F explained that:

Discussion on some mathematics problems such as "how many functions which are both even and odd can we find?" helped me to have better KCT and perspective about the even and odd functions. Regarding this problem some lecturers argued that there is only one even and odd function, but other lecturers believed that although we have only a rule for this function by considering different symmetric domain with respect to the origin we can introduce infinite number of even and odd functions. This discussion was enjoyable for me because before this study my idea was there is only one both even and odd function.

Problem-solving is the key component of mathematics learning and teaching but many of educators do not distinguish between mathematics exercise and mathematics problem. They think every mathematics task is a mathematics problem. This study helped them to learn new things regarding the problem solving approach. Therefore, they provide better mathematical materials in their lessons based on the problem-solving approach. At the beginning of this study, lecturer F described the mathematics problem as follows although later she found what its concept is.

I think every task discusses in the classroom to improve students' learning is considered as a mathematics exercise but mathematics problem refers to the application of mathematics in the real world.

Teaching method

During this research, mathematics lecturers shared their knowledge and experiences on the topic of even and odd functions through collaborative work. All lecturers said that in the LS program, they learned suitable ways of teaching for even and odd functions through discussion with colleagues thus they can transfer this topic to students easier and deeper. They explained that they learned many new methods and techniques that change their views of teaching to use better materials and methods in their classes. Lecturer E explained that:

I have very little experience in teaching mathematics and I am very happy to have joined in this research, because I have learned valuable knowledge about the concepts and methods of teaching even and odd functions.

They believed that one of the best advantages of LS is related to the importance of problem-solving activities in teaching (Bradshaw & Hazell, 2017). Lecturers said LS improved their knowledge in content and pedagogy thus they have better performance in their teaching to improve the ability of students in learning mathematics conceptually. Lecturer C explained:

Traditional teaching method is not suitable for students' learning thus I need to design something different and new in my teaching. LS helps me to find new interest methods to transfer some mathematical concepts to students easier. In this educational method, I consider more details in different parts of my teaching especially to connect different mathematical concepts. Also, in this method, lecturers design the mathematics materials and methods of teaching collaboratively so LS approach is very useful for students learning. For instance, the

application of even function to find the range of absolute functions was exciting for me because I didn't experience this concept before.

Students' activity

In LS, students engage with some activities like problem-solving in the class thus they learn the underlying theory behind the mathematics materials. They can improve in mathematics learning by applying the definitions, formulas, methods and theorems in problem-solving. The lecturers said LS changed their views of teaching and by using rich research lesson and suitable classroom activities, the ability of learners increase in problem-solving especially when they work in groups (Sofroniou & Poutos, 2016). In this research lesson, lecturers focused on problem solving activities through student-centered method to clarify and support how students' previous knowledge was identified and used to prepare the learning activities and to make explicit ways in which the participants actively participate in the constructing mathematical concepts and problem-solving competencies. Lecturer E explained that:

> *In the LS method, students discuss about different activities with suitable methods individually or in group so they have better skills in problem-solving. Thus they will become more self-confident in mathematics learning. For example, through discussion about the activity "find a both even and odd function" students learn new techniques and methods about even and odd functions because they engage with many methods for solving it such as drawing a function graph or finding a rule for this function that satisfies conditions of even and odd functions.*

Lecturers' content and pedagogical content knowledge as identified through observations

The researcher summarized the findings on the even and odd functions based on the observation of discussions among lecturers. These materials show how lecturers improved their knowledge in content and pedagogy especially about SCK and KCT.

In this study, lecturers and the researcher tried to discuss on the even and odd functions in details carefully based on the Mathematics 1 textbook. In the first discussion meeting, the researcher asked lecturers, "Is the function $f(x) = x^2$ with $-3 \leq x \leq 2$ an even function?". The answer of three lecturers was even function since it satisfies the rule $f(-x) = f(x)$. A lecturer answered I only discuss about functions with real numbers domain in my classes. Other lecturers explained that this function is not even because the vertical axis is not its symmetrical axis of the function graph. Through this discussion, lecturers found that the definitions of even and odd functions were ambiguous in the Mathematics 1 textbook. In fact, based on the textbook definitions of even and odd functions, lecturers comprehend different concepts. The textbook definition of even function is as follows:

"A function f is said to be even if and only if $f(-x) = f(x)$ for all x."

It seems that this definition is not clear because some lecturers only focused on the condition $f(-x) = f(x)$ to find whether $y = f(x)$ an even function is. Whereas, we know an even function has a symmetrical domain with respect to the origin. Therefore, lecturers decided to improve the definition of even function as the following:

The function g is considered as an even function if satisfies conditions as follows:

    a.   The function g should has symmetrical domain around the origin

    b. $\forall\, x \in D_g, g(-x) = g(x)$

Similarly, they explained that the odd function k satisfies conditions as follows:

    a.   Domain k must be symmetric around the origin

    b. $\forall\, x \in D_k, k(-x) = -k(x)$

According to the definitions of even and odd functions in the Mathematics 1 textbook, some lecturers only checked the conditions $k(-x) = k(x)$ and $k(-x) = -k(x)$ to determine the even and odd functions respectively. For instance, based on the textbook definition, $k(x) = \sqrt{x} + \sqrt{-x}$ is even because $k(-x) = \sqrt{-x} + \sqrt{x} = \sqrt{x} + \sqrt{-x} = k(x)$. But based on the second definition, $D_k = \{0\}$ and $k = \{(0,0)\}$, thus the function k is both odd and even. So, the lecturers found that the first condition is so important for both definitions of even and odd functions. Three lecturers had serious problem regarding the definitions of even and odd functions. Therefore, they transferred this concept to students superficially. Collaborative work in LS program helped them to improve their content knowledge on the odd and even functions.

The following problem that was created by lecturers in a discussion meeting is so helpful for learners to understand better the concept of odd and even functions. This example improved the KCS among lecturers to prepare appropriate materials for students.

    Example: Give an example of a function for each part.

a.   Odd   b. Even   c. Both odd and even   d. Neither odd nor even

The lecturers argued for part (c), assume that the function h is both even and odd, thus D_h is symmetric around the point $(0,0)$ and we have:

$$\left.\begin{array}{l} h(-x) = h(x) \\ h(-x) = -h(x) \end{array}\right\} \Rightarrow h(x) = -h(x) \Rightarrow 2h(x) = 0 \Rightarrow h(x) = 0.$$

Therefore, the function $h(x) = 0$ is both even and odd. There were different ideas about the number of both even and odd functions among lecturers. Some of them described that there is only one both even and odd function. Other lecturers believed that the numbers of both even and odd functions are infinite. Finally, during this discussion, they concluded that there is only one rule for the function g but if we consider different domains for this function, there would be many answers. For instance, the function $f = \{(-1,0), (0,0), (1,0)\}$ is both even and odd. This discussion helped lecturers to have same idea regarding this problem.

This problem that suggested by a lecturer for the research lesson is suitable to discuss in teaching because it improves mathematical argument among students although there is a mistake in this problem.

Problem: Show that

   a. The sum between two even functions is an even function.

   b. The sum between two odd functions is an odd function.

   c. The product between two odd functions is an even function.

   d. The product between an even function and an odd function is an odd function.

Some lecturers confirmed different parts of this problem by using some examples, instead of logical proofs. For example, for part (c), we know $f(x) = x$ and $g(x) = x^3$ are odd functions, thus $(fg)(x) = f(x)g(x) = x.x^3 = x^4$ is an even function. This method is not logical in mathematics teaching for foundation level students. After discussion about the importance of underlying theory behind the even and odd functions in the process of teaching, they suggested the logical argument as $D_{fg} = D_f \cap D_g$ is symmetric around $(0,0)$ because both D_f and D_g are symmetric around the origin, and it means the first condition for definition of even and odd functions is met. Also, the second condition of definition is held, because functions f and g are odd with the properties $f(-x) = -f(x)$ and $g(-x) = -g(x)$ and hence $(fg)(-x) = f(-x)g(-x) = (-(f(x))(-(g(x)) = f(x)g(x) = (fg)(x)$. Therefore, the property $(fg)(-x) = (fg)(x)$ shows the function fg is an even function. The lecturers were agreeing that this argument helps learners to enhance their skills in problem-solving.

The lecturers pointed out the important issue that students only retain these characteristics without understanding the concept. They emphasized on the conceptually teaching of these properties with logical proofs instead of simply transfer them to students through memorization method. Therefore, the lecturers were suggested to use logical proof for all parts of this problem. For instance, for the logical argument of part (d), assume that $g \neq 0$ and $h \neq 0$ are odd and even respectively. It can easily be seen that the domain of the function gh is symmetric around the origin, because $D_{gh} = D_g \cap D_h$ and both D_g and D_h are symmetric around the origin. Based on the two properties $h(-x) = h(x)$ and $g(-x) = -g(x)$, we have $(gh)(-x) = g(-x)h(-x) = (-g(x))h(x) = -g(x)h(x) = -(gh)(x)$. Therefore, gh is an odd function. As respect to this logical argument, students can find that it is always the product between two non-zero even and odd functions is odd. During discussion about the above problem, the lecturers found that all parts of this problem were rejected by considering the functions $g(x) = 0$ and $h(x) = 0$. Because the sum of these two functions $(g + h)(x) = 0$ is both even and odd. Therefore, they decided to improve this problem as the following:

Problem: Show that

a. The sum between two non-zero even functions is an even function.

b. The sum between two non-zero odd functions is an odd function.

c. The product between two non-zero odd functions is an even function.

d. The product between a non-zero odd function and a non-zero even function is an odd function.

Similarly, the lecturers produced a lot of problems about the composite of even and odd functions such as

Problem: Prove that if f and g are odd functions then fog is an odd function.

In this session, the lecturers explained that for any arbitrary function k, the function $f(x) = k(x) + k(-x)$ is even and $g(x) = k(x) - k(-x)$ is odd. As a result, they clarified that every function k is the sum of two even and odd functions as follows:

$$k(x) = \frac{k(x) + k(-x)}{2} + \frac{k(x) - k(-x)}{2}$$

In other words, they showed every function $k: R \to R$ can be considered as the sum of two even and odd functions. For example, the above property can be used for the function $f(x) = e^x$ as follows:

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \cos hx + \sin hx.$$

Where, the function $y = \cos hx = \frac{e^x + e^{-x}}{2}$ is even and $y = \sin hx = \frac{e^x - e^{-x}}{2}$ is odd. This problem was attractive for the lecturers because they learned new properties regarding the even and odd functions as SCK.

Discussion about the following problem among lecturers shows an application of the even function in the concept of function range.

Problem: Determine the range of $f(x) = \frac{|x|+1}{|x|+2}$.

The range of this function through routine way of solution was argued among LS group members as follows:

$$y = \frac{|x|+1}{|x|+2} \Rightarrow y(|x| + 2) = |x| + 1 \Rightarrow |x| = \frac{2y-1}{1-y}.$$

Since $|x| \geq 0$, we have $\frac{2y-1}{1-y} \geq 0$. The answer of this inequality is the range of the function f as $R_f = \left[\frac{1}{2}, 1\right)$.

The above solution is related to the CCK of lecturers but during discussion about this problem they learned to solve it using SCK as follows. The y-axis is the symmetrical axis of the graph of even functions. Therefore, we need to discuss about the non-negative real numbers of domain (or non-positive real numbers of domain) to find the range of even functions. If x is a non-positive real number then $|x| = -x$ and the function $f(x) = \frac{|x|+1}{|x|+2}$ becomes $g(x) = \frac{-x+1}{-x+2}$. Figure 1 shows the function graph of g.
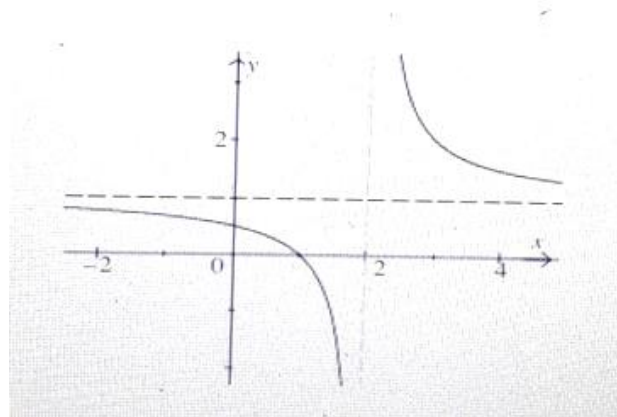

Figure 1. The Function Graph of g

As respect to the Figure 1, the range of the function g is $R_g = \left[\frac{1}{2}, 1\right)$ because only for the non-positive real numbers of domain, the range was determined. Since $g(0) = \frac{0+1}{0+2} = \frac{1}{2}$ thus the range of the function f is $\left[\frac{1}{2}, 1\right)$. Drawing the graph of $f(x) = \frac{|x|+1}{|x|+2}$ is more challenging compare to the graph of $g(x) = \frac{-x+1}{-x+2}$. This discussion shows we can obtain the function range of f using the graph of g. Similarly, for non-negative real number values of domain $f(x) = \frac{|x|+1}{|x|+2}$ becomes $h(x) = \frac{x+1}{x+2}$. Therefore, we can find the range of the function f by drawing the graph of the function h. In fact, we need check only the part of the graph $y = h(x)$ that located on the right side of the y-axis. Discussing this problem helped lecturers to become familiar with the use of the even function in the concept of function range, and increase their SCK and KCS.

Discussion

In this study, a group of mathematics lecturers participated in the LS program. They collaborated on the topic of the odd and even functions to improve their knowledge in content and pedagogy. Based on the lecturers' responses, the LS program was enjoyable and exciting for them because they learned new concepts of content knowledge and methods in teaching. During discussions, the lecturers considered many factors like content knowledge, time management, teaching method, lecturers' materials, students' assessment and student activity to create better materials for students. In fact, the implementation of LS influenced their behaviors to prepare suitable mathematical materials in the research lesson and transfer them to students through better pedagogy. Lecturers discussed about different parts of even and odd functions carefully to produce appropriate output for students. For example, they enhanced the definitions of odd and even functions because the textbook definitions only focused on the algebraic conditions $f(-x) = f(x)$ and

$f(-x) = -f(x)$ to determine even or odd functions respectively. Whereas, three of lecturers understood that the domain of these functions should be symmetric with respect to the origin and enhanced their misconception regarding this issue. Therefore, LS was a strong professional development program in foundation level that improved content and pedagogical content knowledge of lecturers on the topic of even and odd functions. The lecturers prepared high-quality tasks and materials in the research lesson, to emphasize on the theory behind the mathematical concepts instead of superficial teaching methods (Lewis, 2016). So, the best advantage of LS was to highlight the conceptually learning among lecturers and students because learning the concepts of even and odd functions through memorization method is so boring. For example, this study showed the lecturers can determine the range of some kinds of functions easily, by using the properties of even function. Therefore, they linked two different mathematical concepts and improved their teaching knowledge especially on SCK and KCT.

Observations of the lecturers' activities about the even and odd functions showed that lecturers shared their content and pedagogical content knowledge through collaborative work to prepare suitable materials and interesting instruction for students. They learned to create suitable problems for students according to their abilities. At the beginning of this study, the majority of lecturers explained that they teach exactly the topics, concepts, and examples of the textbook because it is difficult to design a new problem for foundation level due to the complexity of the content. Regarding this issue, the researcher described for the lecturers, every small change in the given problems that create a challenge for students considered as a new problem. Therefore, the lecturers learned to how engage their students with new problems based on their abilities especially through collaborative work. For example, for the given problem "is the function $g(x) = \sqrt{x} + \sqrt{-x}$ even?" they replaced the function g with the functions $h(x) = \sqrt{3x} + \sqrt{-3x}$ and $k(x) = \sqrt{x} + \sqrt{-x} + 2$ to produce new problems. Discussion about these two functions showed although $D_h = D_k = \{0\}$ the function h is a both even and odd function whereas the function k is an even function. The lecturers found that in foundation level, learning the underlying theory of mathematical concepts is so important in teaching thus the concepts should not be memorized in mathematics teaching and learning. They agreed that this teaching method improve the ability of students in problem-solving and prepare learners to continue their studies at the university level with suitable basic knowledge. In this program, the lecturers more focused on the logical proofs of the concepts of odd and even functions. For instance, in problem "Prove that the sum and difference between two non-zero odd functions are odd function" the lecturers emphasized on the mathematical arguments instead of simply confirm it by some examples.

In each subtopic, the lecturers tried to consider appropriate activities to enhance the power of learners in problem-solving. For instance, the lecturers discussed about the number of both even and odd functions. Some of them argued based on the definitions in the textbook and explained that there is only one both even and odd function with the rule $f(x) = 0$. Other lecturers believed that the domain of even and odd functions should be symmetric around the origin so there are many functions with this property. Therefore, not only the lecturers were agreeing to change the definitions of even and odd functions but also they summarized that there is a unique rule for both odd and even function. If we limited the domain of this function, there would be a lot of answers. For instance, the function $f = \{(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)\}$ is a both even and odd function.

In this program, the lecturers realized that pedagogical content knowledge has an effective role in the efficient and creative teaching of even and odd functions. They discussed about the suitable methods to transfer the materials on the even and odd functions to students in the better ways. Thus they improved their KCT.

Experienced instructors by having appropriate pedagogical content knowledge are fully acquainted with the educational theories, individual differences of students, strengths and weaknesses of students, and appropriate curriculum materials, and hence, they use suitable teaching methods to better convey mathematical concepts to students (Mohd et al., 2021). Therefore, teamwork in teaching mathematics prepares an ideal opportunity for novice educators to enhance their knowledge in teaching.

Furthermore, the members of LS group taught in a same foundation center and they know the level of students' ability in problem-solving. Thus, they tried to provide appropriate problem-solving activities in this research lesson and hence KCS were enhanced among them. For example, considering the problems "is the function $f(x) = x\sqrt{9 - x^2}$ odd?", "show that the function $g(x) = [\sin 3x - 2\cos 3x]$ is neither even nor odd" and "show that the function $g(x) = \sqrt{[x] + [-x] + 1}$ is an even function" as students activity in the class help students to link different concepts such as radical functions, trigonometric functions and integral part functions.

This study showed mathematics lecturers improved their pedagogical content knowledge and content knowledge about the topic of even and odd functions. This findings are in line with the researches by Capone et al. (2022), Alamri (2020), Akiba et al. (2018), Nishimura et al. (2018), and Takahashi and McDougal (2016).

Conclusion

In foundation centers, mathematics lecturers have limited professional development programs as compared to mathematics teachers in school level. The programs are very much needed so as to improve the subject matter knowledge, abilities and the quality of teaching among lecturers. This is especially necessary for lecturers who are less experienced and those who have applied mathematics or pure mathematics degrees because they do not have much exposure to pedagogy and educational theories and may lack pedagogy. LS is deemed to be the solution in enhancing the pedagogical content knowledge and content knowledge of lecturers in teaching effectively in the foundation center. The LS approach engages the lecturers with collaborative discussions about the even and odd functions and improve their understanding of mathematical concepts. This educational approach also provides an interesting way for lecturers to share their experience and knowledge regarding content knowledge and pedagogical content knowledge. Although this study was about the even and odd functions lecturers can experience LS approach about different mathematical topics in the future. The findings of this research show that LS approach is very useful for increasing the knowledge of pre-university teachers, despite the fact that this method is mostly used in primary, middle and high schools. Therefore, the findings of this research is in line with Cheng and Yee (2011) that explained we can use LS at any educational level for every subject.

Recommendations

The researcher recommended that, future researchers use LS program by testing various teaching methodologies in different foundation centers in other states of Malaysia, different mathematical topics for foundation level, ability of students in problem-solving and impact of LS on lecturers motivation in mathematics.

Limitations

This study involved seven mathematics lecturers from a Malaysian public university and the researcher as participants of the LS group. The findings of this research are based on the data collected from interviews with mathematics lecturers and observations of their discussions. Therefore, the accuracy and truthfulness of the data are also based on the quality of participants' responses and participations.

Disclosure statement

The author of this research declares that he has no conflicts of interest.

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