Research Article https://doi.org/10.12973/ejmse.2.2.85

European Journal of Mathematics and Science Education

Volume 2, Issue 2, 85 - 100.

ISSN: 2694-2003

http://www.ejmse.com/

Exploring Zimbabwean A-Level Mathematics Learners' Understanding of the Determinant Concept

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Received: July 6, 2021 ▪ Revised: October 22, 2021 ▪ Accepted: November 24, 2021

Abstract: Learners bring prior knowledge to their learning environments. This prior knowledge is said to have an effect on how they encode and later retrieve new information learned. This research aimed at exploring 'A' level mathematics learners' understanding of the determinant concept of 3×3 matrices. A problem-solving approach was used to determine learners' conceptions and errors made in calculating the determinant. To identify the conceptions; a paper and pencil test, learner interviews, and learner questionnaires were used. Ten learners participated in the research and purposive sampling was used to select learners who are doing the syllabus 6042/2 Zimbabwe School Examination Council (ZIMSEC). Data was analyzed qualitatively through an analysis of each learners' problem-solving performance where common themes were identified amongst the learners' work. Results from the themes showed that Advanced level learners faced some challenges in calculating the determinant of 3×3 matrices. Learners were having challenges with the place signs used in 3×3 matrices, especially when using the method of cofactors. The findings reveal that learners had low levels of engagement with the concepts and the abstract nature of the concepts was the major source of these challenges. The study recommends that; teachers should engage learners for lifelong learning and apply some mathematical definitions in real-world problems. Teachers should address the issues raised in this research during the teaching and learning process. In addition, teachers should engage learners more through seminars where learners get to mingle with others from other schools.

Keywords: *Linear algebra, matrix, and determinant, understanding.*

To cite this article: Chagwiza, C. J., Sunzuma, G., & Mutambara, L. H. N. (2021). Exploring Zimbabwean a-level mathematics learners' understanding of the determinant concept. *European Journal of Mathematics and Science Education, 2*(2), 85-100. https://doi.org/10.12973/ejmse.2.2.85

Introduction

The teaching and learning of mathematics entail activities that are informed by views about what is essential or valuable; so, the teaching and learning of mathematics are also concerned about values, concerning nurturing the well-being or on the contrary diminishing it. Mathematics education research then focuses on how mathematics is taught and learned in the classroom. the nature of teaching and learning of mathematics, the more the nature of mathematics knowledge and learners 'understanding is understood, the better the educators could plan effectively for teaching approaches and tasks as demanded by the 21st-century paradigm that would improve the understanding and learning of mathematics. In the past, a great deal of the work done in the teaching and learning of mathematics focused on methods of improving mathematics teaching both in tertiary institutions and schools. From the researches done, the common feeling was that learners have challenges in conceptualizing mathematics ideas. Naidoo (2011) discovered that most learners depend on procedures and rules (procedural understanding) when solving mathematical problems. They lack the conceptual aspect, and do not like mathematics, and are demoralized. They end up having monophobia of the concept learned. If one lacks conceptual understanding, he/she tends to struggle in solving unfamiliar problems. Difficulties learners encounter at 'A' level with the learning of mathematics if not checked, will impact negatively on their learning of the subject at tertiary institutions.

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Literature review

Tall (2008) who developed the idea of 'three worlds of mathematics' asserts that every single world develops in complexity and learners follow diverse routes through their growth in mathematics. Obstacles occur in the different routes followed by learners during their growth that call for prior thoughts to be reexamined and remodeled. Tall (2008) goes on to state that "advanced mathematical thinking is characterized by two important components: precise mathematical definitions (including the statement of axioms) and logical deductions of theorems based upon them" (p. 495). Interest in mathematics education has increased during the previous years. Many scholars such as Aygor and Ozdag (2012), Naidoo (2011), and Ndlovu and Brijlall (2019) have all carried out studies centering on instructional approaches as well innovative thinking in mathematics teaching and learning.

Dubinsky (1991), the father of Action Process Object Schema (APOS) theory provides a theoretical framework that focuses on mental constructions that can explain the processes involved in the learning of advanced mathematics. Prior to that, Tall and Vinner (1981) came up with the conceptual framework that explains the construction of knowledge in terms of concept image and concept definition, three worlds of mathematical thinking. As alluded to earlier, there is a vast literature in the area of matrices, but there is limited literature on learners' understanding of the determinant of 3×3 matrices. The current study intends to add more knowledge to understanding, teaching, and learning of matrices.

In the past years, several researchers have focused on learners' difficulties in solving linear algebra problems such as systems of linear equations without tackling the main problematic area, the calculation of determinants of 3×3 matrices. Through studying the systems of simultaneous linear equations, mathematical matrices were developed, of which determinant is an integral part of their solutions. The first famous illustration of the use of matrix methods in solving simultaneous equations is found in the Chinese text (300 BC and AD200) with nine chapters of mathematics art. The development of linear algebra originated from determinants, values linked to a square matrix (3×3 matrices, for example) as propounded by Leibnitz during the late 17th century. The notion of solving systems of linear equations using determinants was developed by Cramer. Most researches done on linear algebra was carried out at tertiary institutions, nothing much was done at the secondary school level, and this has motivated the researchers to explore 'A' level mathematics learners' understanding of the determinant of 3×3 matrices. The challenges learners face range from failing to allocate +/- signs when using the cofactor method, labeling entries in a matrix for example $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Ndlovu and Brijlall (2015) identified that Some pre-service mathematics teachers confused the notations A^t and A^{-1}. Instead of calculating the transpose of A, they calculated the inverse of A. In another study Aygor and Ozdag (2012) found that students confused the determinant operations with the matrix operations while solving problems involving determinants. Students were asked to show that $det\ (B)\ =\ -det\ (A)$, when two rows of matrix A are interchanged to give matrix B. The results revealed that instead of interchanging the rows of matrix A, the students changed the sign of matrix A.

Kazunga and Bansilal (2017) conducted a study about the misconceptions displayed by students when solving examples in matrices. In their finding, they indicated that many of the in-service secondary school teachers misused the use of the equal sign as an operator symbol. For example, when asked to find the transpose of a matrix, they wrote A = A^t. The students here displayed a lack of understanding with regards to a matrix and transpose and also, considering that these participants were already practicing teachers at secondary level, this is really a cause for concern.

Linear algebra in particular the area of matrices is a vital field of study in mathematics. A lot of complex expressions of automated and electrical systems might be appropriately solved through stating them in "determinant form".

According to Stewart and Thomas (2007), a few investigations in the area of matrix determinants have been conducted. Studies that have been carried out showed that the abstract nature of linear algebra and abstraction contributed to the learners' difficulties in the determinant calculation. Furthermore, the topic on matrices at the 'A' level comprises countless concepts that learners have scant (if any) experience in solving certain problems (Hillel, 2000). Transformation using 2×2 matrices, which is one of the concepts under matrices, makes learners develop advanced abstractions whilst they struggle to comprehend numerous new concepts.

Dubinsky (1997) is of the view that priority has to be given to epistemologically examine the concepts that learners have challenges in intellectualizing as well as to outline methods of reasoning in linear algebra. It is from this background of challenges faced by learners in calculating determinants of n×n matrices that the researcher decided to study/explore learners' understanding of the determinant of 3×3 matrices. The Zimbabwe School Examination Council (ZIMSEC), (2015) syllabus code 6042/2 requires learners to calculate the determinant of 3×3 matrices. Learners' background of calculating the determinant of 2×2 matrices i.e., if M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$|M|$ = ad – bc from ordinary level though limited, is of importance at 'A' level. Exploring various ways in calculating determinants of 3×3 matrices is of importance. Changing learner's way of thinking and problem solving is one of the key goals in repairing mathematics and science misconceptions, especially in matrices. Sierpinska (2000) stated that in spite of efforts concentrated on the improvement of the teaching and learning of matrix determinants, learners continued to face difficulties in comprehending linear algebra concepts. Several studies have directed their attention to learners' difficulties as they move from arithmetic to linear algebra. The researchers then focused on the discussion on misconceptions about the determinant calculation of 3×3 matrices.

Thus, the main goal of this study is to establish learner's conceptualization of matrix determinants of (3×3 matrices) and also assess their levels of conceptual understanding in calculating determinants of 3×3 matrices. The learners were exposed to the following methods of calculating determinants during the learning sessions: Dodgson's condensation method; Sarrus's rule; basketweave method for determinants; Chio's condensation method; Striangle's rule; cofactor expansion; and Hajrizaj's method.

The theoretical framework used in the current study is the APOS theory. The APOS theory was used by Kazunga and Bansilal (2020), who investigated in-service teachers' use of determinant and inverse matrices to solve systems of equations in a linear algebra course. Across the group of participants, it was revealed that very few of them were operating at the object level.

Ndlovu and Brijlall (2019) investigated pre-service mathematics teachers' mental constructions while using Cramer's rule. The objective of their study was to describe the type of mental constructions associated with the type of the solution set of equations as well as the meaning of parameters in the solution of equations involving parametric coefficients. Such mental constructions take place in the Action-Process-Object-Schema (APOS) theory. The researchers recommended further research focusing on mental constructions of Cramer's rule.

Mutambara and Bansilal (2019) carried out exploratory research at a university comprising in-service mathematics teachers' comprehending of vector subspaces. The APOS theory was used to guide their study. The study showed that most of the in-service teachers were operating at the action level, or even below the action level, notions of the essential concepts of binary operations and sets hindered their involvement in higher-level concepts of proving axioms.

Ndlovu and Brijlall (2016), conducted a study on pre-service mathematics teachers' mental constructions when solving determinants. The view that the mental constructions that are made by learners whilst learning mathematics concepts could improve the teaching approach guided their study. The findings showed a disagreement on the pre-service teachers' mental constructions and the preliminary genetic decompositions. In addition, most of the pre-service teachers were at the action/ process level, except for only a few who were operating at an object level. The study also revealed that most of the teachers were able to carry out processes successfully, without an understanding of the concept, implying that such teachers' knowledge of the determinant concept is procedural. This study is guided by the following research question:

1. What challenges are faced by A-level mathematics learners when solving problems involving determinants of 3×3 matrices?

2. What levels of conceptual understanding are displayed by advanced level mathematics learners in calculating determinants of 3×3 matrices?

The vital role played by the determinant in various linear algebra concepts and its wide application in other areas of mathematics and science (Kazunga & Bansilal, 2020), motivated the researchers immensely to carry out this study. This study was guided by the APOS theory as explained by Asiala et al. (2004), Dubinsky (1991), and Dubinsky and McDonald (2001). APOS theory begins with the proposition that mathematics knowledge comprises a learner's predisposition to deal with apparent mathematics problem circumstances through the construction of mental *actions*, *processes*, and *objects* as well as consolidating them in *schemas* to understand the circumstances and solve the problems. The APOS theory has remained beneficial in endeavoring to comprehend learners' learning of a wide array of topics in discrete mathematics, statistics, abstract algebra, calculus to mention only a few. The theory is grounded on Piaget's belief that learners learn mathematics through the application of certain mental processes to construct detailed mental structures that are used to solve problems related to the corresponding circumstances (Piaget, 1970). APOS theory asserts that:

> An action conception is a transformation of a mathematical object by individuals according to an explicit algorithm which is conceived as externally driven. As individuals reflect on their actions, they can interiorize them into a process. Each step of a transformation may be described or reflected upon without actually performing it. An object conception is constructed when a person reflects on actions applied to a particular process as a totality or encapsulates it. A mathematical schema is considered as a collection of action, process and object conceptions, and other previously constructed schemas, which are synthesized to form mathematical structures utilized in problem situations (Dubinsky & McDonald, 2001).

An *action* is an alteration of objects perceived by the learner as principally exterior and as required from the memory and a step by step process on how the operation is performed, for instance calculating the determinant of 3x3 matrices when using the cofactor method, one has to put signs on a selected row, for example, using the first row of the given matrix B where

$$B = \begin{pmatrix} +b_{11} & -b_{12} & +b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

i.e., determinant B $=+b_{11} \begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix} - b_{12} \begin{pmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{pmatrix} + b_{13} \begin{pmatrix} b_{21} & b_{22} \\ b_{31} & ba_{32} \end{pmatrix}$

If an action is recurrent such that the learner reflects on it, it is possible to make an internal mental construction known as a process in which the learner might think of carrying out a similar type of action without the exterior stimuli. The learner might think of acting out a procedure, but in reality, do not carry it out, and hence might think about reversing it and combining it with other procedures. If the learner is conscious of the procedure as a totality and understands that alterations can act on it, then it becomes an *object.*

A learner's assortment of actions, processes, objects are known as the *schema.* Some schemas that are connected through some common ideologies in forming a framework in the learner's mind that could be brought to endure on a problem concerning a concept, for example, determinant calculation. The framework should be comprehensible so that it provides a clear method of defining which phenomena might be in the range of the schema and might not. APOS theory deliberates that the entire mathematics objects might be exemplified in terms of actions, processes, objects, and schemas. A genetic decomposition can provide a depiction of a possible path for a learners' concept formation. APOS has several advantages, one of which is that, in APOS, topics are designed around the steps in mental constructions, where learners are actively involved in the learning process rather than just mere spectators.

Methodology

Research design

A research design is a guideline within which a choice about data collection methods has to be made. Cohen et al. (2007) defined it as a method for gathering empirical data with which to test a hypothesis or develop a theory. This study adopted a qualitative research design. This approach will help the data to be more reliable and highly accurate as it combines the primary and secondary data. Interviews, questionnaires, and a written test were implemented in this study. The researchers designed a questionnaire for learners so that they could express their views on determinants of 3x3 matrices. The questionnaire included both open and closed questions. Interviews were conducted with the learners as a follow-up to the written test that was administered. These interviews aim to seek further classification of ideas on learners' understanding of the determinant of 3×3 matrices. This gave the learners another chance to explain further their earlier responses to questions on the questionnaire and written test.

Sampling and Research ethics

Purposive sampling was used to select the ten learners who were doing 'A' level at the selected school. Ethical issues such as informed consent, confidentiality, consent to gather data, and the protection of respondents were taken into consideration in the current study. Approval to carry out the study was granted by the relevant school administrators. Before handing out the questionnaires, the researcher informed the participants about the objective and processes of the study. Participants were informed that participation was voluntary and consent was sought from those who were willing to participate. The participants signed the informed consent slips. The participants were ensured that they could ask questions freely. The participants were also ensured that there would be no names on the questionnaire and that letters say A, B, C, D, etc. would be used to identify respondents. To avoid biased answers from the participants, they were guaranteed that if their confidentiality was endangered, all the records would be burnt. Computer data was protected by a password as a way of keeping the data safe for future reference purposes.

Validity and reliability:

Research scholars concur that a good research instrument should meet the criterion of reliability and validity. Mainly, internal validity is concerned with the congruence of the research findings with the reality. Stability and reliability will be achieved by giving out questionnaires to the representative sample of 'A' level mathematics learners. On the whole, to boost the internal validity of the research data and instruments, the researcher applied: triangulation, long-term observation at research site. In order to strengthen the validity of evaluation data and findings, the researcher collected data through several sources: questionnaires, interviews and a written test. Gathering data through one technique can be questionable, biased and weak. However, collecting information from a variety of sources and with a variety of techniques can confirm findings. Therefore, if we obtain the same results, we can become sure that the data are valid.

Analysis of items from the written test;

The test tool comprised three major questions. Each question had sub-questions, and all in all eight questions were analyzed. The questions were crafted so that learners would be placed in the APOS level they were operating in. Question 1 was set on the order of matrices. This was a recall question, which required learners to show their understanding of the order of naming matrices, a concept taught at O-level mathematics. Question 2, involved a bit of calculation. Under this question, learners were expected to demonstrate their routine eloquence for example to follow steps applied when calculating the determinant of a 3 x 3 matrix using their desired method. Lastly, Question 3 was of a higher order, it was a bit abstract. Under this question, learners were required to make use of problem-solving skills, display their comprehending of the association among concepts, as well as the application of knowledge and processes in explaining the meaning of the concept. The analysis of learners' answers for every single question as well as their extracts were presented in the tables. Although the marks were distributed as a way of grouping the responses, the analysis focused on each learner's procedural and conceptual eloquence. To ensure grade reliability of the test, items were scrutinised before administering them so that the test functions well and is free from bias.

Results

Order of matrix

This section focused on learners' mental construction in finding the order of a given matrix. Question 1 is shown below as item 1. Item 1 requires learners to determine the order of a matrix (of which learners would have done at O-level). The order of a matrix is determined by the number of rows (m) and the number of columns (n), giving an $m\times n$ order. For a square matrix with the same number of rows and columns, the order is $m\times m$.

Item 1:

Identify the order of the matrix below.

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Answer…………………………… Explain……………………………… (2)

Item 1 was designed to try and check whether learners are operating at the ***action*** level of the concept of order of a matrix.

Table 1. Learners' responses for item 1 according to APOS level (action).

APOS Level	N	Action
Number of responses	2	8

Two learners (20%) operated below the action level of the APOS theory and failed to attain the expectation of the preliminary genetic decomposition. The learners failed to provide any response to item 1. Eight (80%) of the learners responded to item 1 that showed that they operated at the action level.

Responses at the action level

Figure 1 below gives the response of learner H who operated at the action level.

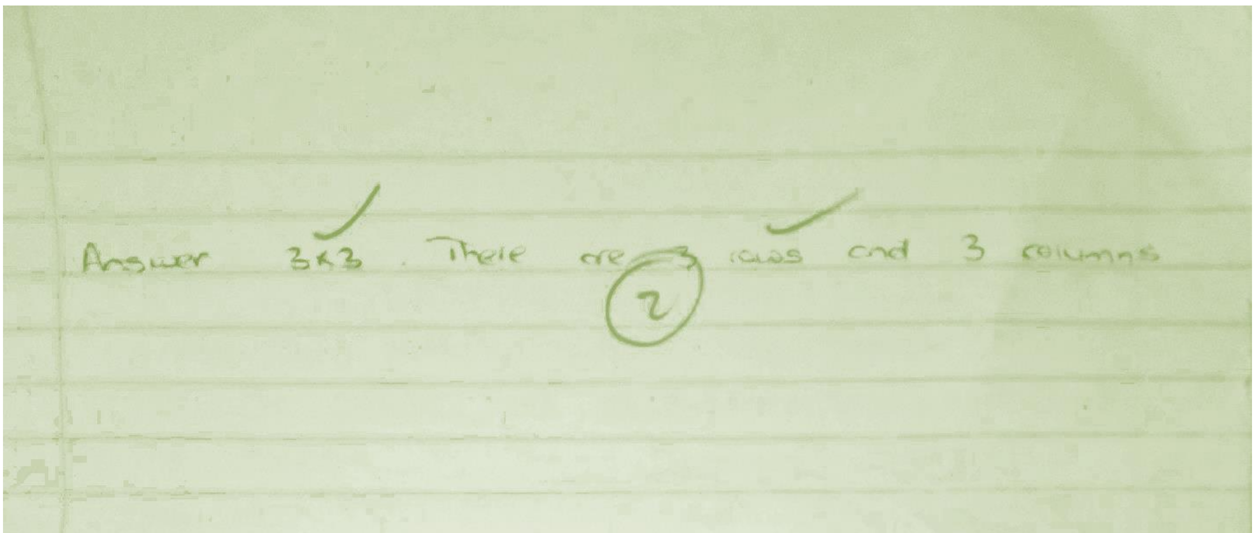

Figure 1. Learner H's response.

Learner H is one of the learners who correctly identified the order of matrix B and provided a correct explanation. Learner C was among the 2 learners who failed to give the order of the matrix. Figure 2 below shows the response of learner C.

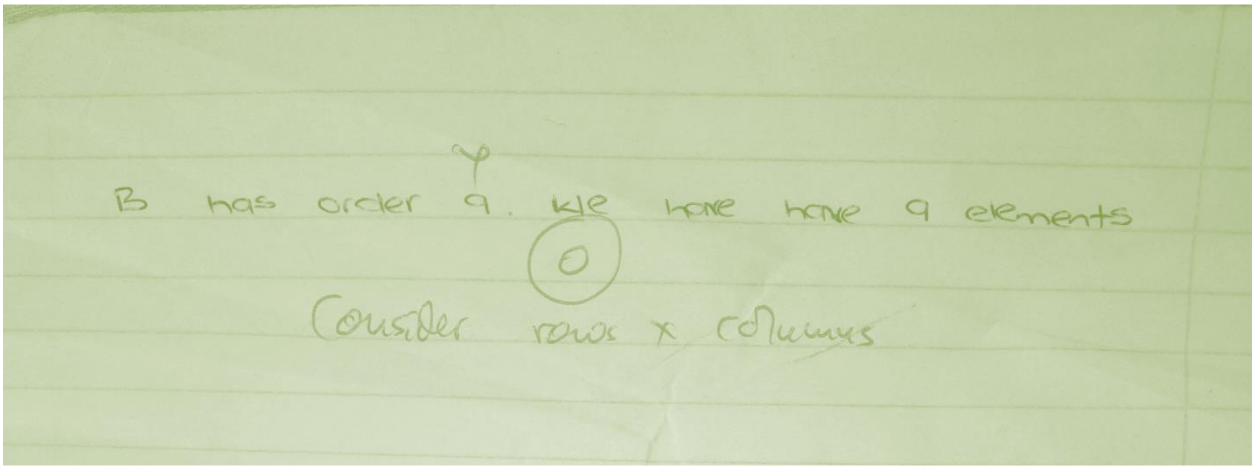

Figure 2. Learner C's response.

From the response by learner C, it is possible that by using × to indicate the order during lessons and in textbooks, the learner might have interpreted it to mean multiplication and that is why the learner simplified by giving 9 as the answer. This would mean that notation has to be explained. This has huge consequences for learners' comprehending of the matrix structure, and this might cause future learning barriers in the understanding of other related concepts if no proper sufficient attention is given.

Matrix transpose, computation of matrices and determinants.

In the written test, Question 2 comprised of 3 sub-questions. This analysis labels these sub-questions as previously done under Question 1, as items. These 3 sub-questions will be presented as items 2 to 4. Item 2 required learners' knowledge of determining a matrix transpose.

Item 2.

Given A = $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$. Determine A^T.

The purpose of item 2 was to check whether were operating at the action level on the concept of matrix transpose.

Table 2. Learners' responses for item 2 according to APOS Level (action).

APOS Level	**N**	**Action**
Number of responses	4	6

Four learners (40%) operated below the action level on item 2 as demanded by APOS theory. They failed to attain the necessary expectations. Six, (60%) of the learners responded to item 2 which indicated that they operated at the actio**n** level. Figure 3 below shows the response of learner I who was amongst the 4 learners operating below the action level. Learner I instead of determining A^t, she tried to find A^{-1}. The notations A^T and A^{-1} were greatly confused here by the learner. The response of learner I is given in the figure below.

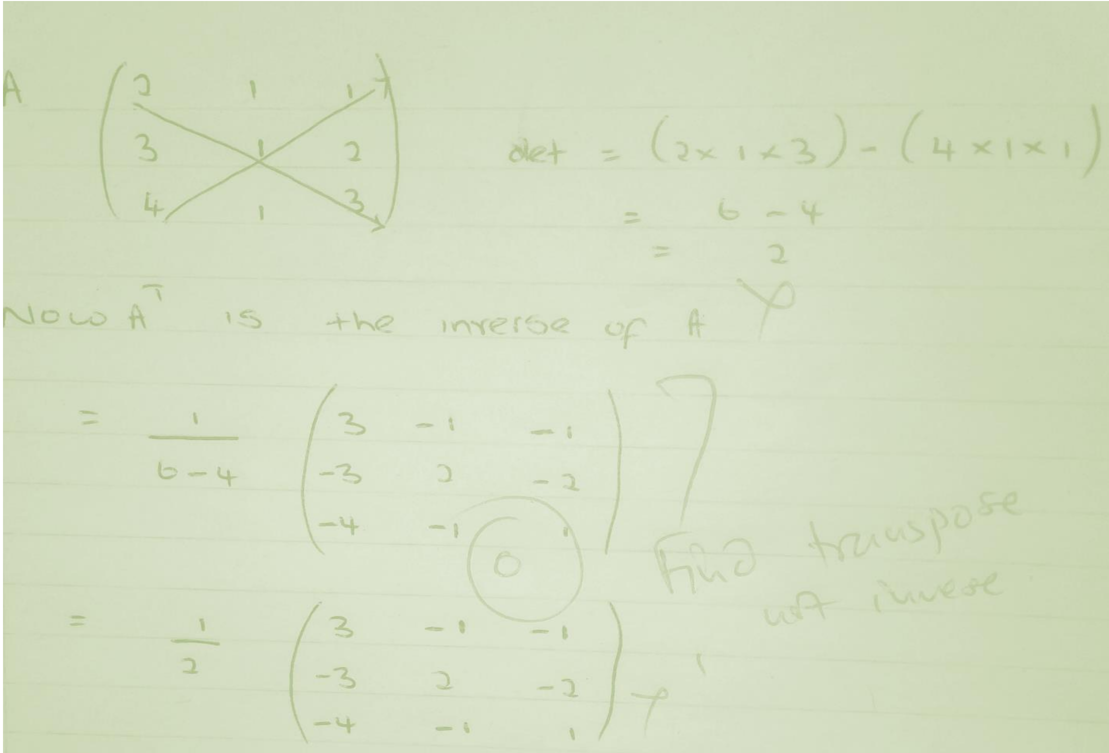

Figure 3. Learner I's response.

From the response above, the learner missed it. The learners could have learned by rote the procedures for calculating the inverse of a 2×2 matrix but did not understand how to apply it to 3×3 matrices. The learner might have learned the concepts as isolated facts. She failed to see the interrelationship between concepts. The learner ended up confusing the two, transpose and inverse. This means that the learner has not developed the action conception of the transpose of a matrix. Six learners (60%) were operating on the action level as they made the required mental construction.

Learner E was one of the learners operating at action level and gave the following response:

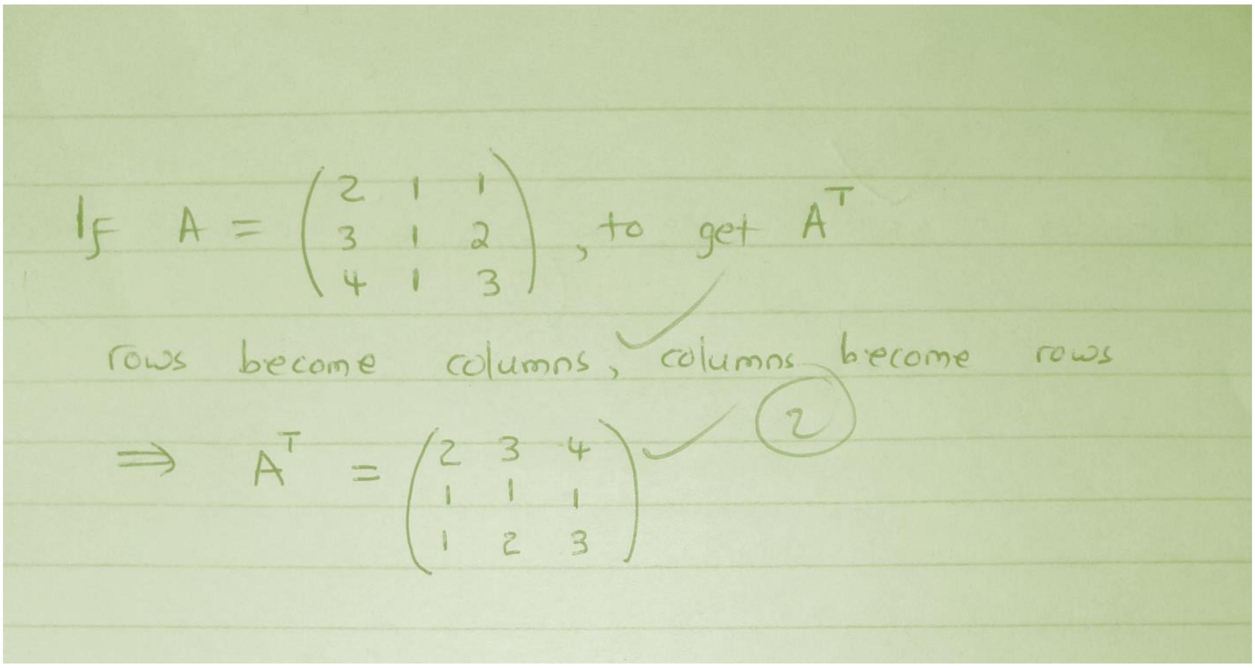

Figure 4. Learner E's response.

From the response above, the learner showed comprehension of the transpose concept and gave a complete response in a series of steps. The learner was operating at the action stage.

Item 3

It focused on learners' understanding of matrix multiplication. The purpose of item 3 was to find out whether the learners were operating at the process level of the matrix product.

$$\text{Let } A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}. \text{ Find AB.}$$

Table 3. Learners' responses for item 3 according to APOS level (process).

APOS Level	N	Process
Number of responses	7	3

Seven learners (70%) operated below the process level of the APOS theory and failed to attain the expectation of the preliminary genetic decomposition. They failed to perform matrix multiplication of 3×3 matrices.

Responses at the process level

Figure 5 gives the solution of learner A. The solution of learner A displayed procedural fluency and competence in determining the product of matrices and manipulation of signs. The process conception had fully developed. Learner A's response:

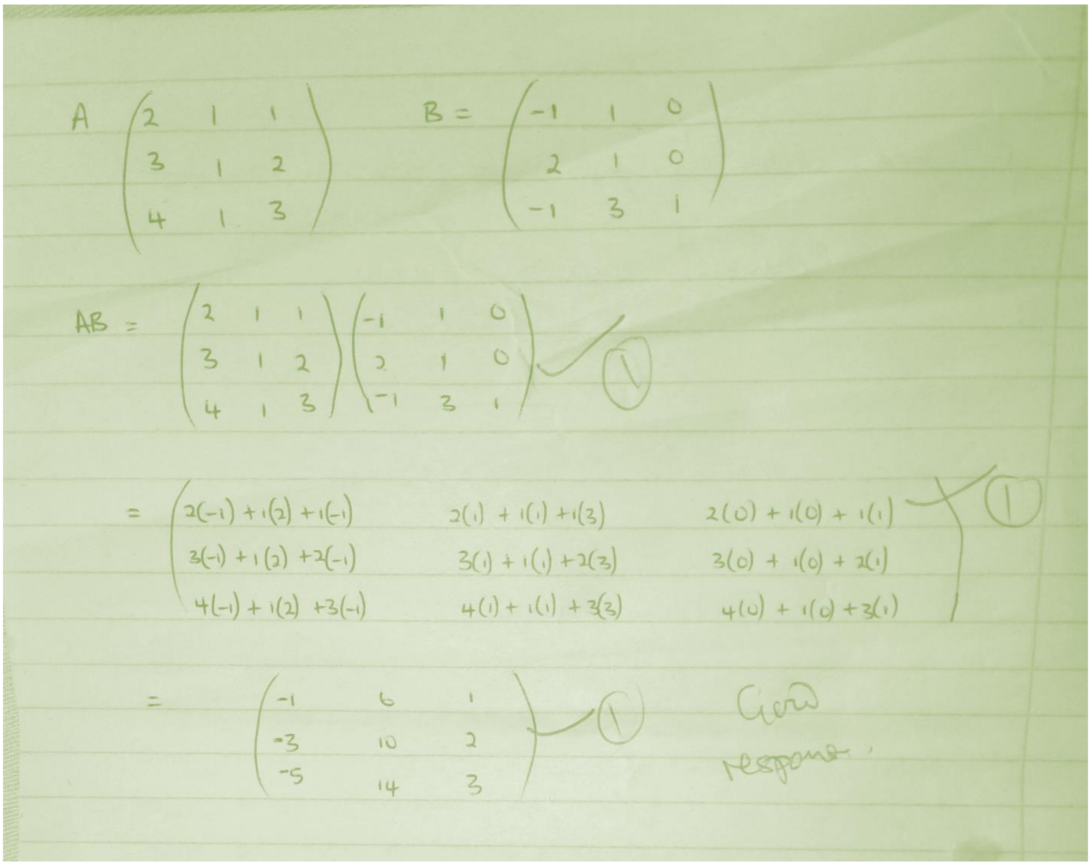

Figure 5. Learner A's response.

From the above response by learner A, the response clearly shows that the learner was operating at the process level as the learner could easily deal with positive ($\pm$) and negative signs to give a comprehensive response. This shows that the learners' actions have been interiorized into the process level.

The responses of the learners in the 70% showed that they were operating at the action level of the matrix product, not the process conception. They correctly interpreted the question, and used correct techniques, but could not manipulate the signs, which showed their limited knowledge of computational skills. The learners failed to deal with directed numbers, that is $+ \times + = +$, some ended up giving $+ \times + = -$, which is a careless mistake. When making such mistakes, it meant that ***process*** conception had not developed, as their responses contained errors.

Item 4

Item 4 aimed at exploring learners' knowledge of evaluating and application of determinants at the process level.

Find the determinant of M = $\begin{pmatrix} 2 & -2 & 1 \\ 3 & 1 & 3 \\ 4 & 2 & -1 \end{pmatrix}$

Table 4. Learners' responses for item 4 according to APOS level.

APOS level	**N**	**Process**
Number of responses	7	3

It was challenging to note that most learners failed to give complete and correct solutions. Only 3 learners (30%) gave complete and correct solutions. These learners operated at the process level in APOS terms and had as well developed the object conception of the determinant of a matrix.

Learner C was among the 7 learners who showed some knowledge of calculating the determinant but failed to perform the processes correctly when using the cofactor method. The response of learner C is as follows;

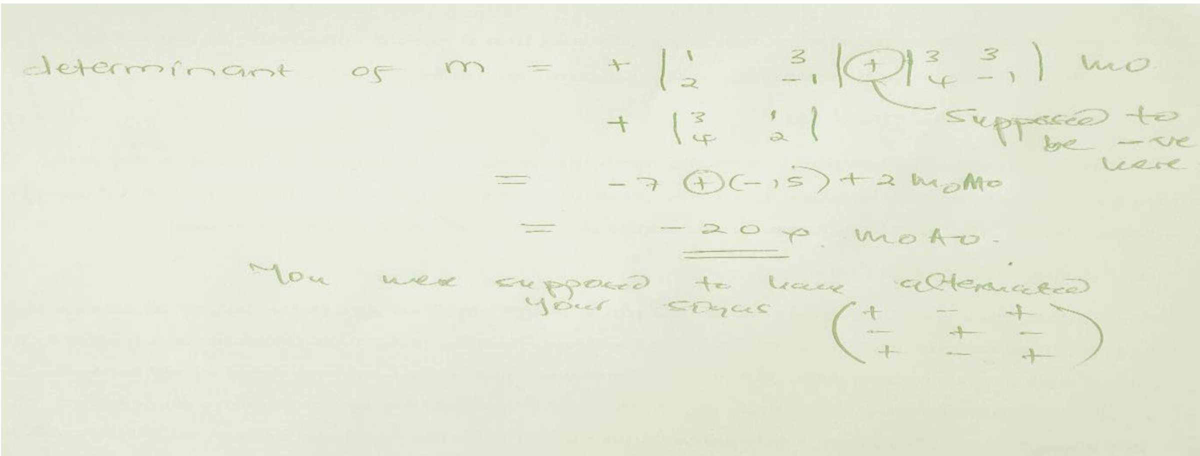

Figure 6. Learner C's response

Learner C tried to use the method of cofactors. Instead of alternating the signs in the first row, he ended up using the (+) sign throughout, and this ended up affecting his solution. Failure to alternate the signs seemed to be a barrier in understanding the concept of the determinant and this hindered the student in developing the necessary concepts beyond the action level.

Learner A was among the 3 learners in category 3 who gave a complete and correct solution as follows;

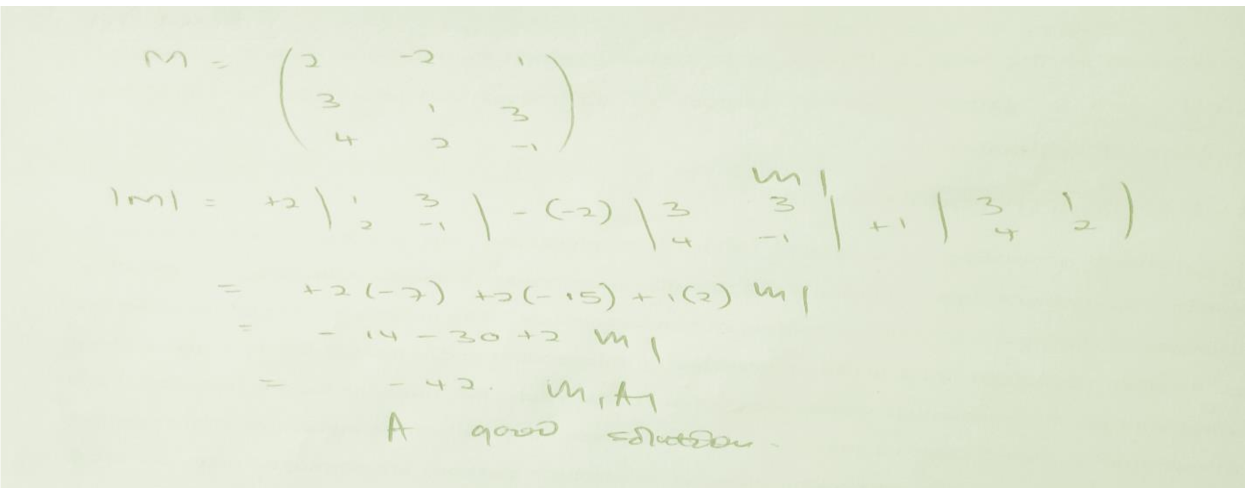

Figure 7. Learner A's response.

The response of learner A showed that the candidate had constructed the corrected concept image of the concept of the determinant of 3×3 matrices. The learner was indeed operating beyond the action stage that is at the ***process*** level as the learner did manage to get the correct solution.

However, some interviews were done to explore further learners' understanding of the determinant concept of 3×3 matrices. The interview follows the written work of the learners' tests. The researchers noticed that some of the learners were not able to finish the given tasks. The learners could start the problem and proceed with one or two steps and abruptly stop before getting the final answer. The researcher asked learner I, the reason for not completing her tasks and the learner replied……

Learner I: "……. I had forgotten the steps Sir and I got stuck".

Booth and Koedinger (2008) posit that many of the mistakes were a result of learners' lack of knowledge or confidence in the problem-solving process.

Kazunga and Bansilal(2018) asserts that when expanding any row or column to compute the matrix determinant, the signs of the coefficients of the minors alternate according to the sign array that follows

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Considering learner C's response figure 6, the researcher asked learner C the reason why he failed to use the place signs on the matrix on cofactors.

Learner C: "……I tend to forget vital concepts under examination conditions, I end up confusing the signs".

The researcher proceeded to ask learner C the reasons for failing to come up with the correct determinant. The interview proceedings were as follows.

Interviewer: What were you doing here on this matrix?

Learner C: I was trying to place the signs on each entry so that I could get the determinant.

Interviewer: How were you placing the signs? Is there a method you were following?

Learner C: I forgot the method, but it involves putting signs, that is the reason why I ended up placing (+) signs throughout.

Interviewer: Why didn't you use other methods of calculating the determinant?

Learner C: I only mastered the method of cofactors, now I remember, I don't know any other method.

Interviewer: Have you ever heard of the Sarrus rule?

Learner C: No

Interviewer: What could be the source of your challenges?

Learner C: The teacher rushed the concepts; I didn't hear anything.

The above interview indicates that the learner was aware that the method of cofactors involves placing signs. However, the learner failed to interchange, or alternate the signs. The learner lacked a conceptual understanding of the concept.

The other issue which is the source of problems encountered by learners is the issue of attending remedial lessons. This was raised by learner G. Learner G raised the issue during the interview when the researcher asked if the learner was attending remedial lessons. Learner G was one of the learners who operated at the object stage of the theory of APOS. The interview proceeded as follows;

Interviewer: You did manage to calculate the determinant correctly, which method were you using here?

Learner G: Our teacher taught us several methods to calculate the determinant of 3x3 matrices. I prefer the cofactor method.

Interviewer: Why do you prefer the cofactor method? Which other methods were you taught?

Learner G: The cofactor method is a bit easier for me plus it is shorter. I remember our teacher telling us about the Sarrus rule, triangle method. I have forgotten the other methods.

Interviewer: Suppose you have encountered challenges in calculating the determinant of 3×3 matrices, do you attend any remedial lessons with the concerned teacher?

Learner G: No I will try to ask my fellow learners.

Interviewer: Why? Are you afraid of your teacher?

Learner G: I am not afraid, the remedial lessons are not conducted for free. Teachers demand payment of which I cannot afford.

Interviewer: What if you fail to get enough help from your colleagues?

Learner G: I usually ask my teacher in class during lesson time.

Interviewer: Are there other learners attending remedial lessons? And how much are they paying?

Learner G: Yes, several learners are attending the lessons. On the issue of payment, I'm not sure how much they are paying.

Interviewer: Since you do not attend remedial lessons, do you sometimes ask the teacher during your spare time?

Learner G: Our teachers use the staff room, they do not have offices, so asking in the staffroom with other teachers seated there, is a bit scary, and I just don't feel comfortable plus you won't understand anything, it would be noisy.

Learners indicating challenges in calculating the determinant of 3x3 matrices:

Out of the 10 learners who took part in the research, 2 learners (20%) faced no challenges when calculating the determinant. 80% of the learners said they faced challenges. Learner A failed /forgot the steps done when calculating the determinants. Learner B asserted that the triangle method and Chio's condensation method were confusing and hard to grasp. This is evidence that learners lack understanding of the determinant of 3×3 matrices. Mathematical understanding could be at an instrumental level (applying rules) or relational (knowing what to do and why) for the 2 learners who faced no challenges. Egodawatte (2009) asserts that conceptions of the determinant of the 3x3 matrix can be developed through the teacher's prior involvements as well as through activities in which the definitions of concept were tested in mathematics teaching and learning. Therefore, some of the sources of conceptions of the determinant of 3x3 matrices are passed on from their teachers. Most of the learners (70%) only remembered the method of cofactors, out of the several available methods, a signal of procedural knowledge without a conceptual comprehending of the determinant concept.

Matrix product and inverse.

The analysis of Question 3 presented as items 5 and 6, aimed at exploring and describing the nature of learners' knowledge of the relationship of matrix product and square matrices.

Item 5.

3.1 Suppose C and D are matrices with CD and DC defined. Explain whether CD and DC are square matrices?

Item 5 intended to place learners' responses at the APOS process stage. The frequency of learners' responses for item 5 is shown in table 5.

Table 5. Learners' responses for item 5 according to APOS level

APOS Level	**N**	**Process**
Number of responses	8	2

Most learners experienced difficulties in interpreting this item as shown by the numbers in the table. Only 20% of the learners managed to provide a supporting argument that was clear and correct. 80% of the learners could not do that. Learners failed to realize that this question was abstract. Learners' reliance on rules seemed to be a problem here. Most learners opted for number grabbing as shown by the response below. Learner F gave the following response;

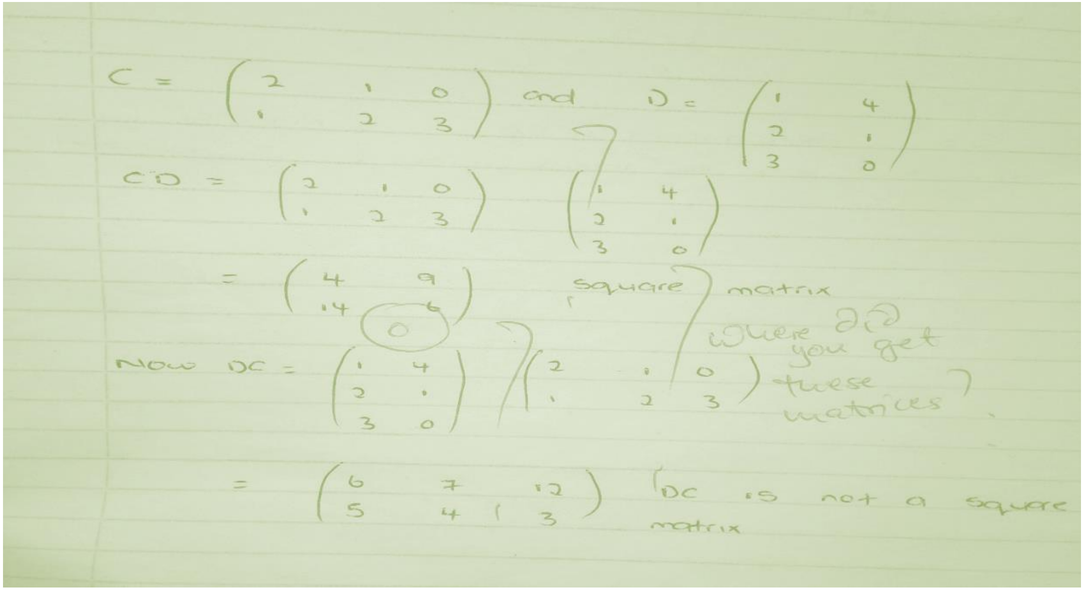

Figure 8. Learner F's response.

Where learner F got the matrices and how the learner came up with numbers is a mystery. The learner also failed to multiply DC properly, evidence that the learner had not achieved the *action* conception of matrix multiplication. The response by the learner indicated that the learner did not understand the question and as a result, for the sake of writing down something, the learner opted for number grabbing. This kind of response indicates that the learner had not interiorized the actions of matrix multiplication into a process.

Two learners operated ate the process level conception of matrix multiplication because of their correct and complete answers. Learner A gave the following response.

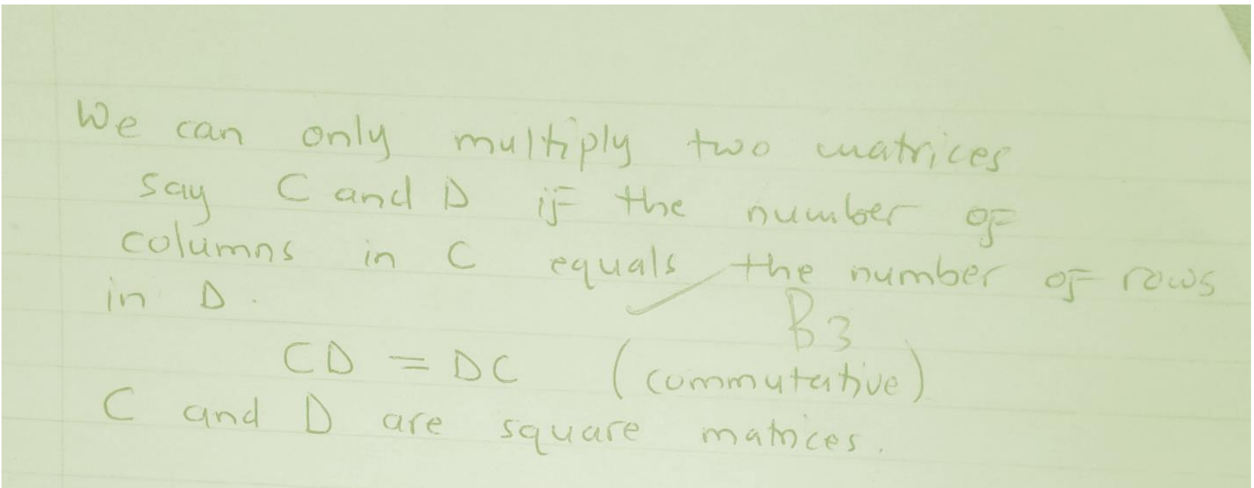

Figure 9. Learner A's response.

From the presentation of learner A, it is clear that the action of finding the matrix product had been interiorized into a process. The learner understood the question. The learner brought out the idea of commutative, which is matrix multiplication is commutative.

Lastly, on the written test, Item 6 intended to check whether learners had a mathematical understanding of singular as well as non-singular matrices concerning inverses of matrices. Insight was sought into whether learners had a conceptual comprehending the inverse of a matrix. The item intended to check whether learners could be placed on the Object level of APOS theory.

Item 6:

Does the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 2) \\ 4 & 2 & 2 \end{pmatrix}$ have an inverse? If so, what is the inverse? If not, explain why?

Table 6. Learners' responses for item 6 according to APOS level (Object).

APOS Level	**N**	**Object**
Number of responses	8	2

Among the seven learners, 2 learners did not make an effort to answer the question, whilst the other 2, simply stated 'YES'.an element of guessing. The researcher faced with such solutions concluded that in terms of APOS, the action conception had not fully developed in these learners.

3 learners stated 'NO' but without a clear justification. Failing to put down a clear justification showed that the learners had not understood the inverse concept under matrices, worse still O- level matrices.

Two learners showed their encapsulation from the process into an object. They provided clear and correct explanations. Learner G was one of the two learners who operated on object level and provided the following response:

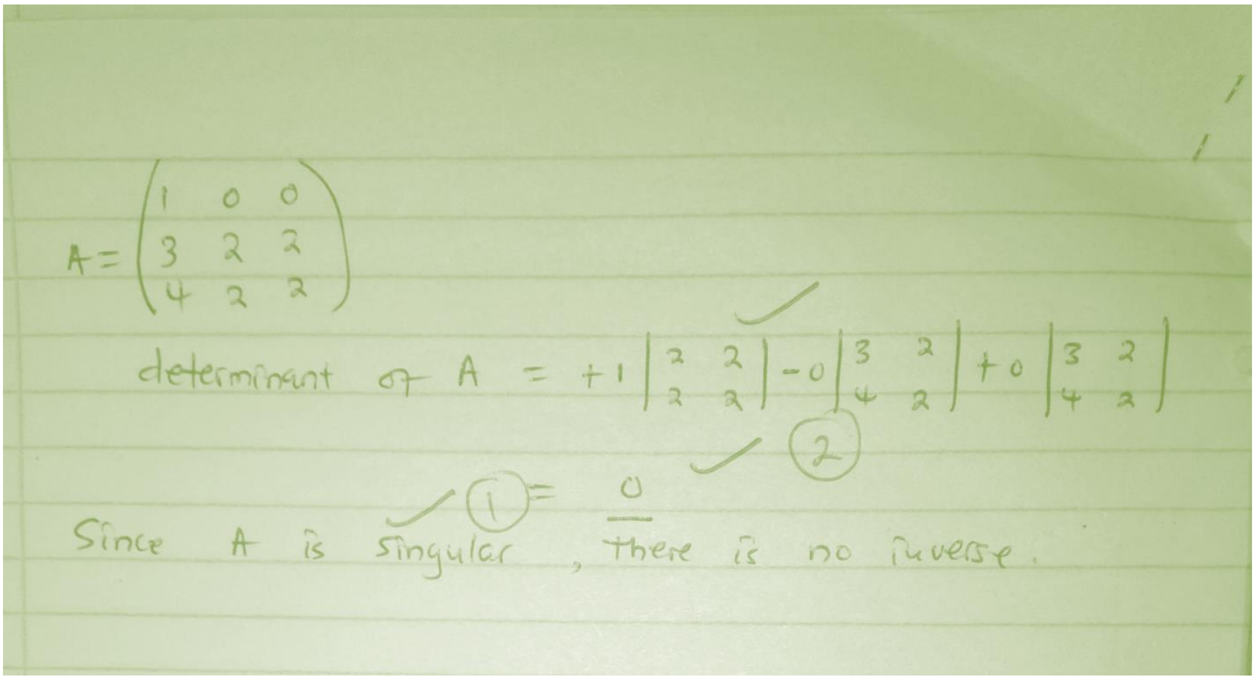

Figure 10. Learner G's response.

Discussion

The response for item 1 on the test was designed requiring learners to display an appreciation of the action level. Many learners at least 8 out of 10, that is 80% managed to attain the action level, with only 2 learners (20%) failing to attain an action level. Learner H was one of the 8 learners who operated at the action level. Learner C failed to attain the action level. The learner misinterpreted × notation on the order of matrices to mean multiplication. The learner gave 9 as the order instead of 3× 3. As noted from this study, most learners operated at the action level. Ndlovu and Brijlall (2019) found out that most pre-service mathematics teachers operated at the action level when using Cramer's rule. In addition, their study showed those who failed to attain the action level displayed a procedural comprehending of Cramer's rule. Mutambara and Bansilal (2019) investigated in-service mathematics teachers' comprehending of vector subspaces discovered that most teachers used reasoning associated with APOS's action level.

Item 2 explored the learner's knowledge of determining the transpose of a matrix. The study showed that 6 out of 10 learners, 60% attained the Action level, with 40% failing to attain the stage, with learner I amongst the 4 learners.

Items 3, 4, and 5 concentrated on learners' understanding of matrix multiplication. The main aim was to check whether the learners were operating at the process level of conception of the matrix product. Out of the 10 learners under this study, only 3 (30%) managed to attain the process level. 70% of the learners made careless mistakes. They even failed to deal with directed numbers. This meant that process conception had not developed. Learner A displayed procedural fluency and competence in determining the product of the two matrices, that is matrix A and matrix B. This was unlike the research done by Kazunga and Bansilal (2017) which detected that the understanding of many learners was at the action level and Process level.

Item 4 was designed at exploring learners' knowledge of evaluating determinants. The item was intended to provide an insight into whether the learners had developed the process conception of evaluating determinants. Only 30% of the learners under this study managed to attain the process level. On item 5, only 20% managed to attain the process level.

Lastly, item 6 was designed to check whether learners had a mathematical understanding of singular and non-singular matrices concerning inverses of matrices. The item was designed to place learners on the Object level. From this study, only 20% of the learners attained the object stage. Learner G was one of the 2 learners who attained the object stage. Kazunga and Bansilal (2020) found out that many of their learners failed to develop suitable mental arrangements at the Object level.

The findings showed that although most learners were able to perform procedural techniques, they were not able to answer questions that required explanations and reasoning. Expression of their thought processes was limited showing an absence of meaning-making amongst the procedures and processes. In most situations, the learners were able to state the answers that lacked supportive statements particularly on item 1 which was on the order of the matrix, indicating a limited conceptual understanding of the learned concept. The findings concurred with earlier results by Siyepu (2013) that learners showed procedural understanding when learning mathematics. In addition, the findings showed that some learners had challenges incorrectly using various terminology and notation, for example, using A^{-1} instead of A^{T}, so, they were not able to construct the essential mental constructions.

The findings clearly showed that most of the learners faced challenges in calculating the determinant of the given 3×3 matrices. Only 20% managed to give complete responses, with 80% making either error when calculating the determinant or failing to calculate the determinant. Learners fail to use their schemas and coordinate them with schemas they would have constructed in other knowledge domains. As alluded to earlier, the results revealed that the main reasons for losing marks were algebraic errors, errors in calculating the determinant. Some of the learners could go as far as using the same sign on the method of cofactors rather than alternating the signs. Levels of conceptual understanding displayed by learners were questioned here. The learners could easily be grouped into two, effective problem solvers and ineffective problem solvers (Egodawatte, 2009). Ineffective as well as effective problem solvers tend to make the same amount of mistakes, however, effective problem solvers are capable of examining tactics to detect and correct mistakes. The written test also showed that learners struggled to give clear explanations on the order of matrices. This shows a clear sign of an incomplete construction of mental images of calculating the determinant of 3×3 matrices. Maybe if previous knowledge of calculating determinants of 2×2 matrices had been interiorized, then the situation could have been different and complete mental structures could have been shown. The APOS theory postulates that it is at all times and circumstances that learners would come to comprehend the procedures by interiorizing actions or it is necessitated by the full development of a mathematics concept (Gilmore & Bryant, 2006). Sherman and Bisanz (2007) challenge the above notion in the case of elementary mathematics. If learners come to comprehend the idea of adding as an object through first interiorizing actions and then encapsulating procedures, it may be expected that they might first attain increasing capability at carrying out addition problems (action), and then begin to reverse such actions, and reflecting on them without performing them (process), before finally being capable of performing new actions on the process (object). All in all, learners need to interiorize their actions to solve a problem, that is to calculate the determinant of 3×3 matrices

Conclusions

The research revealed that learners have challenges in finding the determinant of 3×3 matrices. Only 20% of the learners managed to give accurate and complete responses. The Majority of the learners forgot to place correct signs on the method of cofactors with some ending up placing (+) signs throughout instead of the following $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

The research revealed that learners have challenges with basic numerical fluency and algebraic manipulation skills that affected their performance in the matrix algebra items. Algebraic manipulation skills emerged as a major reason for the incorrect responses that were given by learners. Siyepu (2013) calls such kinds of mistakes, slips.

The research also revealed that learners had problems with mathematical notations, especially on the order of a matrix (row × column). Learners took × as a multiplication thereby ending up getting a single number, that is, the order of 2 × 2 matrices was given as 4.

A Conceptual understanding was lacking in most learners. On finding the determinant of a 3×3 matrix given in the test, most learners remembered the method of cofactors but ended up using the same signs throughout instead of alternating the signs. It was also noted that although learners had a background of 2×2 matrices, a lot still needs to be done for the learners to comprehend and understand the concepts of 3×3 matrices. Although some learners struggled in calculating the determinant of the 3×3 matrix, some learners have mastered the concept as they could easily give complete responses. Hence for the learners to understand, they need to develop deeper engagement of object conception and skills which leads to schema development (Kazunga & Bansilal, 2020).

Recommendation

In view of the research findings stated above, the researcher recommended that: CDU (Curriculum Development Unit) should adopt matrix algebra, both 2×2 and 3×3 matrices at O-level mathematics so that learners would not face challenges at 'A' level. The Curriculum designers should also clearly specify several methods to be used in calculating determinants of 3×3 matrices (5 or more). This would give learners a wider choice to choose an easier method. The researcher encourages 'A' level mathematics teachers to conduct seminars so that they share ideas and strategies of teaching determinants of 3×3 matrices. Teachers should explain in detail the concept of the determinant of 3×3 matrices rather than rushing concepts in class. One learner highlighted that in the interview. The learner accused the teacher of rushing the concepts. With that, learners may develop a strong understanding of the concepts rather than just memorizing the concept. There is a need for conceptual rather than procedural knowledge.

Limitations

The study is limited to only one topic at the 'A' level. The study was carried out at only one school with a small sample, therefore the findings of the current study might not be generalized to schools in Zimbabwe.

Authorship Contribution Statement

Chagwiza: Concept and design, analysis of data, completion of the first draft of the article, and article revision. Mutambara: Article drafting, analysis of data, and article revision. Sunzuma: Article drafting, analysis of data, and article revision.

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